In this section we'll study several convergence tests that apply to series with **positive terms**.

THE INTEGRAL TEST

THEOREM 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Sample Problem #1: USING THE INTEGRAL TEST

Apply the integral test to the following series:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2n^3 - 1}$

$f(n) = \dfrac{n^2}{2n^3-1} \Rightarrow f(x) = \dfrac{x^2}{2x^3-1}$

$\dfrac{1}{6}\displaystyle\int_0^{\infty} \frac{6x^2}{2x^3-1}\,dx = \frac{1}{6} \ln|2x^3-1| \Big]_0^{\infty}$

$\dfrac{1}{6} \displaystyle\lim_{b\to\infty} \left[\ln|2b^3-1| - \ln 1\right] \to \infty$

$\boxed{\text{Therefore } \displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2n^3-1} \text{ diverges}}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 2}$

$f(n) = \dfrac{5}{2n^2+2}$

$f(x) = \dfrac{5}{2} \cdot \dfrac{1}{x^2+1}$

$\displaystyle\int_1^{\infty} \frac{5}{2} \cdot \frac{1}{x^2+1}\,dx =$

$= \dfrac{5}{2}\left[\displaystyle\lim_{b\to\infty} \arctan(b) - \arctan(1)\right]$

$= \dfrac{5}{2}\left[\pi/2 - \pi/4\right] = \dfrac{5}{2} \cdot \dfrac{\pi}{4} = \dfrac{5\pi}{8}$

$\boxed{\text{Therefore } \displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+2} \text{ converges}}$

p-SERIES AND HARMONIC SERIES

A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$ is a ***p*-series**, where *p* is a positive constant. For *p=1*, the series $\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots$ is a **harmonic series**. A **general harmonic series** is of the form $\sum_{n=1}^{\infty}\frac{1}{an+b}$.

THEOREM 9.11 Convergence of *p*-Series

The *p*-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$

1. converges if $p > 1$, and
2. diverges if $0 < p \le 1$.

Sample Problem #2: CONVERGENT AND DIVERGENT *p*-series

Discuss the convergence or divergence of:

a) $\underbrace{\text{the harmonic series}}$ b) $\underbrace{\text{the } p\text{-series with } p=3}$

$p=1 \Rightarrow$ diverges $p>1 \Rightarrow$ converges

Sample Problem #3: **TESTING A SERIES FOR CONVERGENCE**

Determine whether the following series converges or diverges: (INTEGRAL TEST)

a) $\sum_{n=2}^{\infty} \frac{5}{n\ln(n)}$

$f(x) = \frac{5}{x\ln x}$

$\int_2^{\infty} \frac{5}{x\ln x}\,dx = 5\ln(\ln x)\Big]_2^{\infty}$

$= \lim_{b\to\infty} 5\ln(\ln b) - 5\ln(\ln 2)$

$= \infty$ $\boxed{\text{Diverges}}$

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n}-2\right)}$

$f(x) = \frac{1}{\sqrt{x}(\sqrt{x}-2)}$

$2\int_1^{\infty} \frac{1}{2\sqrt{x}(\sqrt{x}-2)}\,dx$

$2\lim_{b\to\infty} \left[\ln|\sqrt{x}-2| - \ln 1\right]$

$2(\infty) = \infty$

$\boxed{\text{Diverges}}$

c) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{4n^2+4}$

$f(x) = \frac{\arctan x}{4(x^2+1)}$

$\frac{1}{4}\int_1^{\infty} \frac{\arctan x}{x^2+1}\,dx$

$u = \arctan x$

$\frac{1}{4}\int_{\pi/4}^{\pi/2} u\,du$

$\frac{1}{4}\left(\frac{u^2}{2}\right)\Big]_{\pi/4}^{\pi/2}$

$\frac{1}{8}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right]$

$\frac{1}{8}\left[\frac{3\pi^2}{16}\right] = \frac{3\pi^2}{128}$

$\boxed{\text{Converges}}$

Sample Problem #4: **TESTING A SERIES FOR CONVERGENCE**

Determine whether the following series converges or diverges: (*p*-SERIES TEST)

a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

$$\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$$

$p=1/2$

$\underbrace{0<1/2\le 1}$

$\boxed{\text{Diverges}}$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\left(\sqrt{n}\right)}$

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

$p=3/2$

$\underbrace{\frac{3}{2}>1}$

$\boxed{\text{Converges}}$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1.04}}$

$p=1.04$

$\underbrace{1.04>1}$

$\boxed{\text{Converges}}$

Sample Problem #5:

Find the positive values p for which the series converges:

a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$

$f(x) = \frac{1}{x(\ln x)^p}$

$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$

$u = \ln x$

$\int_{\ln 2}^{\infty} u^{-p} dx = \left. \frac{u^{1-p}}{1-p} \right]_{\ln 2}^{\infty}$

$\lim_{b \to \infty} \left[\frac{b^{1-p} - (\ln 2)^{1-p}}{1-p} \right]$

$1-p \leq 0$

$\boxed{p \geq 1}$

b) $\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

$f(x) = \frac{x}{(1+x^2)^p}$

$\frac{1}{2}\int_1^{\infty} \frac{2x}{(1+x^2)^p} dx = \frac{1}{2} \int_2^{\infty} u^{-p} du$

$u = 1 + x^2$

$\lim_{b \to \infty} \left(\left. \frac{u^{1-p}}{1-p} \right]_2^{\infty} \right)$

$\lim_{b \to \infty} \left[\frac{u^{1-p} - 2^{1-p}}{1-p} \right]$

$1-p \leq 0$

$\boxed{p \geq 1}$