Alternating Series

An alternating series is a series whose terms are alternately positive and negative on **consecutive** terms.

For instance: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ and $-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$

In general, just knowing that $\lim_{n\to\infty} a_n = 0$ tells us very little about the convergence of the series $\sum_{n=1}^{\infty} a_n$; however, it turns out that an alternating series must converge if its terms consistently shrink in size and approach zero!!

Alternating Series Test (AST)

If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges if both of the following conditions are satisfies:

1) $\lim_{n\to\infty} a_n = 0$

2) $\{a_n\}$ is a decreasing (or Non-increasing) sequence; that is, $a_{n+1} \le a_n$ for all $n > k$. for some $k \in \mathbb{Z}$

Note: This does NOT say that if $\lim_{n\to\infty} a_n \neq 0$ the series DIVERGES by the AST. The AST can ONLY be used to prove convergence. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges, but by the ***n*th-term** test NOT the AST.

Example 16:
Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{2n-1}$

$a_n = \frac{n}{2n-1}$; $\lim_{n\to\infty}\frac{n}{2n-1} = \frac{1}{2} \neq 0$

SO THE SERIES DIVERGES BY n^{TH} TERM TEST.

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{\ln(2n)}$

$a_n = \frac{n}{\ln(2n)}$; $\lim_{n\to\infty}\frac{n}{\ln(2n)} = \infty \neq 0$

SO IT DIVERGES BY

(c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

JUST NEED TO PROVE ITS AN ALT. SERIES: $\cos(n\pi) = (-1)^n$

$\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \to a_n = \frac{1}{n}$

$\lim_{n\to\infty}\frac{1}{n} = 0$; $a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$

$\therefore$ SINCE BOTH CONDITIONS ARE MET BY AST, THE SERIES CONVERGES.

(d) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$

$a_n = \frac{1}{n!}$; $\lim_{n\to\infty}\frac{1}{n!} = 0$ AND

$\frac{1}{n!}$ DECREASES

$\therefore$ SINCE BOTH CONDITIONS ARE MET BY THE AST, THE SERIES CONVERGES

(e) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-5)^2+1}$

$a_n = \frac{1}{(n-5)^2+1}$, $\lim_{n\to\infty}\frac{1}{(n-5)^2+1} = 0$

AND $\frac{1}{(n-5)^2+1}$ DECREASES

$\therefore$ SINCE BOTH CONDITIONS ARE MET BY THE AST, THE SERIES CONVERGES.

(f) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}$

$\lim_{n\to\infty}\frac{1}{n} = 0$

$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$

$\therefore$ SINCE BOTH CONDITIONS ARE MET BY THE AST, THE SERIES CONVERGES.

Example 17:
Determine whether the given alternating series converges or diverges. If it converges, determine whether it is absolutely convergent or conditionally convergent.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ → THIS SERIES CONV B/C $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ + $a_n=\frac{1}{\sqrt{n}}>\frac{1}{\sqrt{n+1}}=a_{n+1}$

$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right|=\frac{1}{\sqrt{n}} \Rightarrow p=\frac{1}{2}\le 1$ SO IT DIV.

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ CONV CONDITIONALLY BY THE AST + P-SERIES

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^n}$ $\lim_{n\to\infty}\frac{1}{3^n}=0$ + $\frac{1}{3^n}$ DEC SO IT CONVERGES

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3^n}\right|=\left(\frac{1}{3}\right)^n$ CONV.

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^n}$ CONV. CONDITIONALLY BY THE AST + GEOM SER TEST.

Alternate Series Remainder

Suppose an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is not increasing. If the series has a sum S, then $|R_n| = |S - S_n| \le a_{n+1}$, where S_n is the nth partial sum of the series.

In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the nth partial sum, S_n, and your error will have an absolute value no greater than the first term left off, a_{n+1}. This means $S \in [S_n - R_n, S_n + R_n]$

Example 18:

Approximate the sum $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$ by using its first six terms, and find the error. Use your results to find an interval in which S must lie.

$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} \to .63194 \to \frac{91}{144}$ APPROX SUM

$|R_6| \le |a_7| = \left|\frac{1}{5040}\right| \to \boxed{\frac{1}{5040}}$

REMAINDER

SO $S \in \left[\frac{91}{144} - \frac{1}{5040}, \frac{91}{144} + \frac{1}{5040}\right]$

$S \in [.6317, .6321]$

Example 19:

Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001