11.5 Comparison Tests

As we begin to compile a list of convergent and divergent series, new ones can sometimes be analyzed by comparing them to ones that we already understand.

EXAMPLE 11.5.1 Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converge?

The obvious first approach, based on what we know, is the integral test. Unfortunately, we can't compute the required antiderivative. But looking at the series, it would appear that it must converge, because the terms we are adding are smaller than the terms of a p-series, that is,

$$\frac{1}{n^2 \ln n} < \frac{1}{n^2},$$

when $n \geq 3$. Since adding up the terms $1/n^2$ doesn't get "too big", the new series "should" also converge. Let's make this more precise.

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converges if and only if $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converges—all we've done is dropped the initial term. We know that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges. Looking at two typical partial sums:

$$s_n = \frac{1}{3^2 \ln 3} + \frac{1}{4^2 \ln 4} + \frac{1}{5^2 \ln 5} + \cdots + \frac{1}{n^2 \ln n} < \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} = t_n.$$

Since the p-series converges, say to L, and since the terms are positive, $t_n < L$. Since the terms of the new series are positive, the s_n form an increasing sequence and $s_n < t_n < L$ for all n. Hence the sequence $\{s_n\}$ is bounded and so converges. □

Sometimes, even when the integral test applies, comparison to a known series is easier, so it's generally a good idea to think about doing a comparison before doing the integral test.

EXAMPLE 11.5.2 Does $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converge?

We can't apply the integral test here, because the terms of this series are not decreasing. Just as in the previous example, however,

$$\frac{|\sin n|}{n^2} \le \frac{1}{n^2},$$

because $|\sin n| \le 1$. Once again the partial sums are non-decreasing and bounded above by $\sum 1/n^2 = L$, so the new series converges. □

Like the integral test, the comparison test can be used to show both convergence and divergence. In the case of the integral test, a single calculation will confirm whichever is the case. To use the comparison test we must first have a good idea as to convergence or divergence and pick the sequence for comparison accordingly.

EXAMPLE 11.5.3 Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$ converge?

We observe that the -3 should have little effect compared to the n^2 inside the square root, and therefore guess that the terms are enough like $1/\sqrt{n^2} = 1/n$ that the series should diverge. We attempt to show this by comparison to the harmonic series. We note that

$$\frac{1}{\sqrt{n^2-3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n},$$

so that

$$s_n = \frac{1}{\sqrt{2^2-3}} + \frac{1}{\sqrt{3^2-3}} + \cdots + \frac{1}{\sqrt{n^2-3}} > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = t_n,$$

where t_n is 1 less than the corresponding partial sum of the harmonic series (because we start at $n = 2$ instead of $n = 1$). Since $\lim\limits_{n\to\infty} t_n = \infty$, $\lim\limits_{n\to\infty} s_n = \infty$ as well. □

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series.

EXAMPLE 11.5.4 Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ converge?

Just as in the last example, we guess that this is very much like the harmonic series and so diverges. Unfortunately,

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{n},$$

so we can't compare the series directly to the harmonic series. A little thought leads us to

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{2n},$$

so if $\sum 1/(2n)$ diverges then the given series diverges. But since $\sum 1/(2n) = (1/2)\sum 1/n$, theorem 11.2.2 implies that it does indeed diverge. □

For reference we summarize the comparison test in a theorem.

THEOREM 11.5.5 Suppose that a_n and b_n are non-negative for all n and that $a_n \le b_n$ when $n \ge N$, for some N.

If $\displaystyle\sum_{n=0}^{\infty} b_n$ converges, so does $\displaystyle\sum_{n=0}^{\infty} a_n$.

If $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges, so does $\displaystyle\sum_{n=0}^{\infty} b_n$.

■

Exercises 11.5.

Determine whether the series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+3n+5} \Rightarrow$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5} \Rightarrow$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2-3n-5} \Rightarrow$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+4}{2n^2+3n+5} \Rightarrow$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^2+3n+5} \Rightarrow$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n} \Rightarrow$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3} \Rightarrow$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} \Rightarrow$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+5^n} \Rightarrow$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n} \Rightarrow$