Based on your experience with improper integrals, again, fill in the chart below.

Comparison of Series

Direct Comparison Test (DCT)

If $a_n \geq 0$ and $b_n \geq 0$,

1) If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ CONVERGES.

IF LARGER SERIES CONV THE SMALLER SERIES MUST ALSO CONV

2) If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ DIVERGES.

IF SMALLER SERIES DIV. THE LARGER SERIES MUST ALSO DIV.

NOTE: You must state/show the inequality when stating the conclusion of the test!!

HIGHEST POWER IN NUM + DEN

Example 14:

Determine whether the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

DIVERGES BY n^{TH} TERM TEST SINCE

$\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1 \neq 0$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

FOR ALL $n \geq 1$

$\frac{1}{n^3+1} \leq \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ CONVERGES

SO $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ CONVERGES

(c) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

FOR ALL $n \geq 1$

$\frac{1}{3^n+2} \leq \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ CONVERGES

SO $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ CONVERGES

(d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

FOR ALL $n \geq 4$

$\frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES

SO $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ DIVERGES

(e) $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

$-\left(\frac{1}{2}\right)^n \leq \frac{\cos n}{2^n} \leq \left(\frac{1}{2}\right)^n$

SINCE THIS IS A GEOM. SERIES WHERE $r = \frac{1}{2} < 1$ THIS SERIES

$\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ CONVERGES

(f) $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$

$a_n = \frac{1}{n^4-10}$ $b_n = \frac{1}{n^4}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^4-10)}{1/n^4}$

$= \lim_{n\to\infty} \frac{n^4}{n^4-10} = 1$. SO THE LIMIT COMPARISON TEST APPLY WHEN $C=1$.

SINCE $p=4>1$, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ CONVERGES

~~ALSO~~ ALSO $\sum_{n=1}^{\infty} \frac{1}{n^4-10}$ CONV.

Sometimes the inequalities needed above don't hold or are difficult to show, but you still strongly suspect the result because you recognize a similar series with which to compare it.

> **Limit Comparison Test (LCT)**
>
> If $a_n \geq 0$ and $b_n \geq 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ **or** $\lim_{n\to\infty} \frac{b_n}{a_n} = L$, where L is both finite and positive.
>
> Then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Example 15:
Determine whether the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

$a_n = \frac{1}{3n^2-4n+5}$ $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1/3n^2-4n+5}{1/n^2} \to \frac{n^2}{3n^2-4n+5} = \frac{1}{3} \neq 0$

BY LCT (w/ $c = \frac{1}{3} \neq 0$), THE P-SERIES $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES B/C $p = 2 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$ ALSO CONVERGES

(b) $\sum_{n=1}^{\infty} \frac{n^4 + 10}{4n^5 - n^3 + 7}$

$\cancel{\lim_{n\to\infty}} \frac{a_n}{\cancel{b_n}} = \frac{n^4}{4n^5} \to \frac{1}{4n}$; $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n^4+10)}{4n^5-n^3+7} \to \frac{1}{4} \neq 0$

BY LCT (w/ $c = \frac{1}{4}$) SINCE $\sum_{n=1}^{\infty} \frac{1}{n}$ HARMONIC DIVERGES, $\sum_{n=1}^{\infty} \frac{n^4+10}{4n^5-n^3+7}$ ALSO DIVERGES

(c) $\sum_{n=2}^{\infty} \frac{1}{n^3 - 2}$

$a_n = \frac{1}{n^3-2}$ $b_n = \frac{1}{n^3}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1/n^3-2}{1/n^3} \to \frac{n^3}{n^3-2} = 1$

BY LCT (w/ $c = 1 \neq 0$), THE P-SERIES $\sum_{n=1}^{\infty} \frac{1}{n^3}$ CONVERGES B/C $p = 3 > 1$.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^3-2}$ ALSO CONVERGES

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$

$a_n = \frac{1}{\sqrt{3n-2}}$ $b_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1/\sqrt{3n-2}}{1/\sqrt{n}} \to \frac{\sqrt{n}}{\sqrt{3n-2}} = \frac{1}{\sqrt{3}}$

BY LCT (w/ $c = \frac{1}{\sqrt{3}}$), THE P-SERIES $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES B/C $p = \frac{1}{2} < 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$ ALSO DIVERGES