11.6 Absolute Convergence

Roughly speaking there are two ways for a series to converge: As in the case of $\sum 1/n^2$, the individual terms get small very quickly, so that the sum of all of them stays finite, or, as in the case of $\sum(-1)^{n-1}/n$, the terms don't get small fast enough ($\sum 1/n$ diverges), but a mixture of positive and negative terms provides enough cancellation to keep the sum finite. You might guess from what we've seen that if the terms get small fast enough to do the job, then whether or not some terms are negative and some positive the series converges.

THEOREM 11.6.1 If $\sum_{n=0}^{\infty}|a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

Proof. Note that $0 \le a_n+|a_n| \le 2|a_n|$ so by the comparison test $\sum_{n=0}^{\infty}(a_n+|a_n|)$ converges. Now

$$\sum_{n=0}^{\infty}(a_n+|a_n|) - \sum_{n=0}^{\infty}|a_n| = \sum_{n=0}^{\infty} a_n + |a_n| - |a_n| = \sum_{n=0}^{\infty} a_n$$

converges by theorem 11.2.2. ∎

So given a series $\sum a_n$ with both positive and negative terms, you should first ask whether $\sum|a_n|$ converges. This may be an easier question to answer, because we have tests that apply specifically to series with non-negative terms. If $\sum|a_n|$ converges then you know that $\sum a_n$ converges as well. If $\sum|a_n|$ diverges then it still may be true that $\sum a_n$ converges—you will have to do more work to decide the question. Another way to think of this result is: it is (potentially) easier for $\sum a_n$ to converge than for $\sum|a_n|$ to converge, because the latter series cannot take advantage of cancellation.

If $\sum|a_n|$ converges we say that $\sum a_n$ converges **absolutely**; to say that $\sum a_n$ converges absolutely is to say that any cancellation that happens to come along is not really needed, as the terms already get small so fast that convergence is guaranteed by that alone. If $\sum a_n$ converges but $\sum|a_n|$ does not, we say that $\sum a_n$ converges **conditionally**. For example $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges absolutely, while $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges conditionally.

EXAMPLE 11.6.2 Does $\sum_{n=2}^{\infty}\frac{\sin n}{n^2}$ converge?

In example 11.5.2 we saw that $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$ converges, so the given series converges absolutely. □

EXAMPLE 11.6.3 Does $\sum_{n=0}^{\infty} (-1)^n \frac{3n+4}{2n^2+3n+5}$ converge?

Taking the absolute value, $\sum_{n=0}^{\infty} \frac{3n+4}{2n^2+3n+5}$ diverges by comparison to $\sum_{n=1}^{\infty} \frac{3}{10n}$, so if the series converges it does so conditionally. It is true that $\lim_{n\to\infty} (3n+4)/(2n^2+3n+5) = 0$, so to apply the alternating series test we need to know whether the terms are decreasing. If we let $f(x) = (3x+4)/(2x^2+3x+5)$ then $f'(x) = -(6x^2+16x-3)/(2x^2+3x+5)^2$, and it is not hard to see that this is negative for $x \geq 1$, so the series is decreasing and by the alternating series test it converges. □

Exercises 11.6.

Determine whether each series converges absolutely, converges conditionally, or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n^2+3n+5} \Rightarrow$

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n^2+4}{2n^2+3n+5} \Rightarrow$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n} \Rightarrow$

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n^3} \Rightarrow$

5. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln n} \Rightarrow$

6. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+5^n} \Rightarrow$

7. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+3^n} \Rightarrow$

8. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan n}{n} \Rightarrow$