Ratio and Root Tests

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$
2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$
3. The ratio test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

Series involving expressions that grow very rapidly such as factorials and/or exponential work especially well with the Ratio Test.

Example 20:
Determine if the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{n^2 3^{n+1}}{2^n}$ (c) $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$

$$L = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \Rightarrow \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{2}{n+1} \Rightarrow 0$$

∴ SINCE L=0, THE SERIES CONV. BY RATIO TEST

Root Test

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$
2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$
3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

If the entire rule of sequence can be written as power of n, the Root Test is hard to beat!

Example 21:

(a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n} \right)^n$

Example 22:

Putting it all together. Determine if the following series converge or diverge. Name the test used and the criteria of each test used.

(a) $\sum_{n=1}^{\infty} \frac{1+3n^2+n^3}{4n^3-5n+2}$

(b) $\sum_{n=0}^{\infty} \left(\frac{2}{7} \right)^n$

(c) $\sum_{n=1}^{\infty} \frac{4}{n^3}$

(d) $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$

(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5+5}}$

(f) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$

(g) $\sum_{n=1}^{\infty} \frac{5n^2-6n+3}{n^3-7n+8}$

(h) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$

(i) $\sum_{n=1}^{\infty} \frac{3^n+4}{2^n}$

(j) $\sum_{n=1}^{\infty} \frac{8n^3-6n^5}{12n^4-9n^5}$

(k) $\sum_{n=1}^{\infty} \sqrt{\frac{3n+1}{n^5+2}}$

(l) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n2^n}$

(m) $\sum_{n=1}^{\infty} \left(\frac{2n}{5n-1} \right)^n$

(n) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

(o) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n} \right)^n$

(p) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

22 a) $\lim_{n\to\infty} \frac{n^3}{4n^3} = \frac{1}{4} \neq 0$ So Series Diverges By n^{TH} Term Test

b) $|r| = \left|\frac{2}{7}\right| = \frac{2}{7} < 1$ So Series Converges By GST $\left(\text{Sum} = \frac{1}{1-\frac{2}{7}} \to \frac{7}{5}\right)$

c) $4\sum_{n=1}^{\infty} \frac{1}{n^3} \to p = 3 > 1$ So Series Converges B/c Of p-Series

d) $\lim_{n\to\infty} \left|\frac{(n+1)^2}{5^{n+1}} \cdot \frac{5^n}{n^2}\right| \to \lim_{n\to\infty} \left|\frac{(n^2+2n+1)\cancel{5^n}}{n^2 \cdot 5 \cdot \cancel{5^n}}\right| = \frac{1}{5} < 1$ So Series Converges By Ratio Test

e) Compare w/ $\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$, A Convergent p-Series $\left(p = \frac{5}{3} > 1\right)$; Since $\frac{1}{\sqrt[3]{n^5+5}} \leq \frac{1}{n^{5/3}}$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5+5}}$ Converges By Direct Comparison.

f) Let $f(x) = \frac{1}{x(\ln x)^4}$ Which Is Positive + Cont. For $x \geq 2$ And $f'(x) < 0$

$\int_2^{\infty} \frac{1}{x(\ln x)^4}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}dx$ $\quad \int_{u=\ln 2}^{\infty} \frac{1}{u^4}\,du = -3u^{-3}\Big]_{u=\ln 2}^{\infty} = \frac{+3}{(\ln 2)^3} = \infty$ So Series Converges By Integral Test

g) Compare w/ $\sum_{n=1}^{\infty} \frac{1}{n}$ (The Divergent Harmonic Series); $\lim_{n\to\infty} \left(\frac{5n^2-6n+3}{n^3-7n+8} \cdot \frac{n}{1}\right) = 5 > 0$

So Series Diverges By The Limit Comparison Test

h) Alternating Series; $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ And $\frac{1}{\sqrt{n}}$ Is Decreasing, So Series Converges By AST

i) $= \frac{3}{2} \neq$ So Diverges By nTH Term Test; $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \to r = \frac{3}{2} > 1$ So Diverges By GST; Or

$\frac{3^n+4}{2^n} \geq \frac{3^n}{2^n}$ It Diverges By Direct Comparison Test

j) $\lim_{n\to\infty} \frac{-6n^5}{9n^5} = \frac{-6}{9} \neq 0$ So Series Diverges By n^{TH} Term Test

k) Compare w/ $\sum_{n=1}^{\infty} \sqrt{\frac{1}{n^4}} = \frac{1}{n^2} \to p = 2 > 1$ So A Convergent p-Series; Or $\lim_{n\to\infty} \left(\sqrt{\frac{3n+1}{n^5+2}} \cdot \sqrt{\frac{n^4}{1}}\right) =$

$\sqrt{\lim_{n\to\infty}\left(\frac{3n^5+n^4}{n^5+2}\right)} = \sqrt{3} > 0$ So Series Converges By Limit Comparison Test

l) $L = \lim_{n\to\infty} \frac{\frac{3^n}{(n+1)2^{n+1}}}{\frac{3^{n-1}}{n2^n}} \to \lim_{n\to\infty} \frac{3^n}{3^{n-1}} \cdot \frac{n2^n}{(n+1)2^{n+1}} = \frac{3n}{2(n+1)} = \frac{3}{2} > 1$ So Series Diverges By Ratio Test

m) $L = \lim_{n\to\infty} \left|\left(\frac{2n}{5n-1}\right)^n\right|^{1/n} \to \lim_{n\to\infty} \left|\frac{2n}{5n-1}\right| = \left|\frac{2}{5}\right| = \frac{2}{5} < 1$ So Series Converges By Root Test (Abs. Conv)

n)

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Putting it all together. Determine if the following series converge or diverge. Name the test used and the criteria of each test used.

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(b) $\sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n$

(c) $\sum_{n=1}^{\infty} \frac{4}{n^3}$

(d) $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$

(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5+5}}$

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(n) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

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(p) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

How Do You Know Which Test To Use?

P p-series: Is the series in the form $\frac{1}{n^p}$?

A Alternating series: Does the series alternate? If it does, are the terms getting smaller, and is the nth term 0?

R Ratio Test: Does the series contain things that grow very large as n increases (exponentials or factorials)?

T Telescoping series: Will all but a couple of the terms in the series cancel out?

I Integral Test: Can you easily integrate the expression that defines the series (are Dogs Cussing in Prison?)

N nth Term divergence Test: Is the nth term something other than zero?

G Geometric series: Is the series of the form $\sum_{n=0}^{\infty} ar^n$?

C Comparison Tests: Is the series ***almost*** another kind of series (e.g. p-series or geometric)? Which would be better to use: the Direct or Limit Comparison Test?