THE RATIO TEST

**THEOREM 9.17 Ratio Test**

Let $\Sigma\, a_n$ be a series with nonzero terms.

1. $\Sigma\, a_n$ converges absolutely if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1.$
2. $\Sigma\, a_n$ diverges if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty.$
3. The Ratio Test is inconclusive if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$

**Sample Problem #1:** **USING THE RATIO TEST**

Determine the convergence or divergence of the series:

a) $\sum_{n=0}^{\infty} \frac{4^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \frac{4 \cdot \cancel{4^n}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{4^n}} = \frac{4}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{4}{n+1} = 0 < 1 \Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{4^n}{n!} \text{ converges absolutely}}$$

b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \frac{(n+1)^n}{n^n}$$

$$= \frac{n^n + n \cdot n^{n-1} + \dots}{n^n}$$

$$= 2 + \frac{\cdots}{n^n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} 2 + \frac{\cdots}{n^n} = 2$$

$$2 > 1 \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges}}$$

c) $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^2\, 2^{n+2}}{3^{n+1}}}{\dfrac{n^2\, 2^{n+1}}{3^n}} = \frac{(n^2+2n+1)\, 2\cdot \cancel{2^{n+1}}}{3\cdot \cancel{3^n}} \cdot \frac{\cancel{3^n}}{n^2 \cdot \cancel{2^{n+1}}}$$

$$= \frac{2(n^2+2n+1)}{3n^2}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n^2+4n+2}{3n^2} = 2/3$$

$$\frac{2}{3} < 1 \Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n} \text{ is absolutely convergent}}$$

d) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{\sqrt{n+1}}{n+2}}{\dfrac{\sqrt{n}}{n+1}} = \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n+1}{n+2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \sqrt{1+\frac{1}{n}} \cdot \frac{1+1/n}{1+2/n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt{1+1/n} \cdot \frac{1+1/n}{1+2/n} = 1$$

$\Rightarrow$ The ratio test is inconclusive for $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$