

THE RATIO TEST

THEOREM 9.17 Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
2. $\sum a_n$ diverges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Sample Problem #1: USING THE RATIO TEST

Determine the convergence or divergence of the series:

a) $\sum_{n=0}^{\infty} \frac{4^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \frac{4 \cdot \cancel{4^n}}{(n+1)n!} \cdot \frac{n!}{\cancel{4^n}} = \frac{4}{n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{4}{n+1} = 0 < 1 \Rightarrow \sum_{n=0}^{\infty} \frac{4^n}{n!} \text{ converges absolutely}$$

b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} 2 + \frac{\dots}{n^n} = 2$$

$2 > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges

$$= \frac{(n+1)^n}{n^n}$$

$$= \frac{n^n + n \cdot n^{n-1} + \dots}{n^n}$$

$$= 2 + \frac{\dots}{n^n}$$

c) $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 2^{n+2}}{3^{n+1}} = \frac{(n^2+2n+1) 2 \cdot 2^{n+1}}{3 \cdot 3^n} \cdot \frac{3^n}{n^2 \cdot 2^{n+1}}$$

$$= \frac{2(n^2+2n+1)}{3n^2}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2n^2+4n+2}{3n^2} = \frac{2}{3}$$

$\frac{2}{3} < 1 \Rightarrow \sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ is absolutely convergent

$$d) \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{\sqrt{n+1}}{n+2}}{\frac{\sqrt{n}}{n+1}} = \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n+1}{n+2}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \sqrt{1 + \frac{1}{n}} \cdot \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} \cdot \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} = 1 \Rightarrow \text{The ratio test is inconclusive for } \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$