11.8 Power Series

Recall that we were able to analyze all geometric series "simultaneously" to discover that

$$\sum_{n=0}^{\infty} kx^n = \frac{k}{1-x},$$

if $|x| < 1$, and that the series diverges when $|x| \geq 1$. At the time, we thought of x as an unspecified constant, but we could just as well think of it as a variable, in which case the series

$$\sum_{n=0}^{\infty} kx^n$$

is a function, namely, the function $k/(1-x)$, as long as $|x| < 1$. While $k/(1-x)$ is a reasonably easy function to deal with, the more complicated $\sum kx^n$ does have its attractions: it appears to be an infinite version of one of the simplest function types—a polynomial. This leads naturally to the questions: Do other functions have representations as series? Is there an advantage to viewing them in this way?

The geometric series has a special feature that makes it unlike a typical polynomial—the coefficients of the powers of x are the same, namely k. We will need to allow more general coefficients if we are to get anything other than the geometric series.

DEFINITION 11.8.1 A power series has the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

with the understanding that a_n may depend on n but not on x. □

EXAMPLE 11.8.2 $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is a power series. We can investigate convergence using the ratio test:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \frac{n}{|x|^n} = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x|.$$

Thus when $|x| < 1$ the series converges and when $|x| > 1$ it diverges, leaving only two values in doubt. When $x = 1$ the series is the harmonic series and diverges; when $x = -1$ it is the alternating harmonic series (actually the negative of the usual alternating harmonic series) and converges. Thus, we may think of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ as a function from the interval $[-1, 1)$ to the real numbers. □

A bit of thought reveals that the ratio test applied to a power series will always have the same nice form. In general, we will compute

$$\lim_{n\to\infty} \frac{|a_{n+1}||x|^{n+1}}{|a_n||x|^n} = \lim_{n\to\infty} |x| \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L|x|,$$

assuming that $\lim |a_{n+1}|/|a_n|$ exists. Then the series converges if $L|x| < 1$, that is, if $|x| < 1/L$, and diverges if $|x| > 1/L$. Only the two values $x = \pm 1/L$ require further investigation. Thus the series will definitely define a function on the interval $(-1/L, 1/L)$, and perhaps will extend to one or both endpoints as well. Two special cases deserve mention: if $L = 0$ the limit is 0 no matter what value x takes, so the series converges for all x and the function is defined for all real numbers. If $L = \infty$, then for any non-zero value of x the limit is infinite, so the series converges only when $x = 0$. The value $1/L$ is called the **radius of convergence** of the series, and the interval on which the series converges is the **interval of convergence**.

Consider again the geometric series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Whatever benefits there might be in using the series form of this function are only available to us when x is between -1 and 1. Frequently we can address this shortcoming by modifying the power series slightly. Consider this series:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x+2}{3}\right)^n = \frac{1}{1-\frac{x+2}{3}} = \frac{3}{1-x},$$

because this is just a geometric series with x replaced by $(x+2)/3$. Multiplying both sides by $1/3$ gives

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} = \frac{1}{1-x},$$

the same function as before. For what values of x does this series converge? Since it is a geometric series, we know that it converges when

$$\begin{aligned} |x+2|/3 &< 1 \\ |x+2| &< 3 \\ -3 < x+2 &< 3 \\ -5 < x &< 1. \end{aligned}$$

So we have a series representation for $1/(1-x)$ that works on a larger interval than before, at the expense of a somewhat more complicated series. The endpoints of the interval of convergence now are -5 and 1, but note that they can be more compactly described as -2 ± 3. We say that 3 is the radius of convergence, and we now say that the series is centered at -2.

DEFINITION 11.8.3 A power series centered at a has the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n,$$

with the understanding that a_n may depend on n but not on x. □

Exercises 11.8.

Find the radius and interval of convergence for each series. In exercises 3 and 4, do not attempt to determine whether the endpoints are in the interval of convergence.

1. $\displaystyle\sum_{n=0}^{\infty} nx^n \Rightarrow$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n \Rightarrow$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}(x-2)^n \Rightarrow$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}(x-2)^n \Rightarrow$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)} \Rightarrow$

11.9 Calculus with Power Series

Now we know that some functions can be expressed as power series, which look like infinite polynomials. Since calculus, that is, computation of derivatives and antiderivatives, is easy for polynomials, the obvious question is whether the same is true for infinite series. The answer is yes:

THEOREM 11.9.1 Suppose the power series $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence R. Then

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1},$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1},$$

and these two series have radius of convergence R as well. ■

EXAMPLE 11.9.2 Starting with the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = -\ln|1-x| = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

$$\ln|1-x| = \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1}$$

when $|x| < 1$. The series does not converge when $x = 1$ but does converge when $x = -1$ or $1 - x = 2$. The interval of convergence is $[-1, 1)$, or $0 < 1 - x \leq 2$, so we can use the series to represent $\ln(x)$ when $0 < x \leq 2$. For example

$$\ln(3/2) = \ln(1 - -1/2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \frac{1}{2^{n+1}}$$

and so

$$\ln(3/2) \approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384} + \frac{1}{896} = \frac{909}{2240} \approx 0.406.$$

Because this is an alternating series with decreasing terms, we know that the true value is between $909/2240$ and $909/2240 - 1/2048 = 29053/71680 \approx .4053$, so correct to two decimal places the value is 0.41.

What about $\ln(9/4)$? Since $9/4$ is larger than 2 we cannot use the series directly, but

$$\ln(9/4) = \ln((3/2)^2) = 2\ln(3/2) \approx 0.82,$$

so in fact we get a lot more from this one calculation than first meets the eye. To estimate the true value accurately we actually need to be a bit more careful. When we multiply by two we know that the true value is between 0.8106 and 0.812, so rounded to two decimal places the true value is 0.81. □

Exercises 11.9.

1. Find a series representation for $\ln 2$. $\Rightarrow$
2. Find a power series representation for $1/(1-x)^2$. $\Rightarrow$
3. Find a power series representation for $2/(1-x)^3$. $\Rightarrow$
4. Find a power series representation for $1/(1-x)^3$. What is the radius of convergence? $\Rightarrow$
5. Find a power series representation for $\int \ln(1-x)\,dx$. $\Rightarrow$