WORKSHEET 1 ON POWER SERIES

Work these on **notebook paper**, except for problem 1.

1. Derive the Taylor series formula by filling in the blanks below.

   Let $f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + a_5(x-c)^5 + \ldots + a_n(x-c)^n + \ldots$

   What happens to this series if we let $x = c$?

   $f(c) =$ __________ so $a_0 =$ __________

   Now differentiate $f(x)$ to find $f'(x)$ and $f'(c)$.

   $f'(x) =$

   $f'(c) =$ __________ so $a_1 =$ __________

   Differentiate again, and find $f''(x)$ and $f''(c)$.

   $f''(x) =$

   $f''(c) =$ __________ so $a_2 =$ __________

   Now find $f'''(x)$ and $f'''(c)$.

   $f'''(x) =$

   $f'''(c) =$ __________ so $a_3 =$ __________

   Do you see a pattern? $f^{(n)}(c) =$ ______________ so $a_n =$ ______________

   Now substitute your results into

   $f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + a_5(x-c)^5 + \ldots + a_n(x-c)^n + \ldots$

   $f(x) =$ ____ + ____ $(x-c)$ + _______ $(x-c)^2$ + _______ $(x-c)^3$ + ... + _______ $(x-c)^n + \ldots$

On problem 2, find a Taylor series for $f(x)$ centered at the given value of c. Give the first four nonzero terms and the general term for the series.

2. $f(x) = e^{2x}$, $c = 3$

On problem 3 - 4, find a Taylor series for $f(x)$ centered at the given value of c. Give the first four nonzero terms. (You do not need to give the general term.)

3. $f(x) = \sin\left(2x + \frac{\pi}{3}\right)$, $c = 0$

4. $f(x) = \cos x$, $c = \frac{2\pi}{3}$

On problems 5 – 8, find a Maclaurin series for $f(x)$. Give the first four nonzero terms and the general term for each series.

5. $f(x)=\sin\left(x^3\right)$

6. $f(x)=\dfrac{\cos(3x)}{x}$

7. $f(x)=x^2e^{-x}$

8. $f(x)=\sin^2 x$ $\left(\text{Hint: Use the fact that } \sin^2 x=\dfrac{1-\cos(2x)}{2}.\right)$

Answers

Worksheet 1 on Power Series

1. $a_0 = f(c)$, $a_1 = f'(c)$, $a_2 = \dfrac{f''(c)}{2!}$, $a_c = \dfrac{f'''(c)}{3!}$, $a_n = \dfrac{f^{(n)}(c)}{n!}$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots + \frac{f^{(n)}(c)(x-c)^n}{n!} + \ldots$$

2. $e^6 + 2e^6(x-3) + \dfrac{4e^6(x-3)^2}{2!} + \dfrac{8e^6(x-3)^3}{3!} + \ldots + \dfrac{2^n e^6(x-3)^n}{n!} + \ldots$

3. $\dfrac{\sqrt{3}}{2} + x - \dfrac{2\sqrt{3}x^2}{2!} - \dfrac{4x^3}{3!} + \ldots$

4. $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\left(x - \dfrac{2\pi}{3}\right) + \dfrac{\left(x - \dfrac{2\pi}{3}\right)^2}{2\cdot 2!} + \dfrac{\sqrt{3}\left(x - \dfrac{2\pi}{3}\right)^3}{2\cdot 3!} + \ldots$

5. $x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} - \dfrac{x^{21}}{7!} + \ldots + \dfrac{(-1)^n x^{6n+3}}{(2n+1)!} + \ldots$

6. $\dfrac{1}{x} - \dfrac{9x}{2!} + \dfrac{81x^3}{4!} - \dfrac{729x^5}{6!} + \ldots + \dfrac{(-1)^n 3^{2n} x^{2n-1}}{(2n)!} + \ldots$ where $x \neq 0$

7. $x^2 - x^3 + \dfrac{x^4}{2!} - \dfrac{x^5}{3!} + \ldots + \dfrac{(-1)^n x^{n+2}}{n!} + \ldots$

8. $\dfrac{2x^2}{2!} - \dfrac{8x^4}{4!} + \dfrac{32x^6}{6!} - \dfrac{128x^8}{8!} + \ldots + \dfrac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!} + \ldots$

WORKSHEET 2 ON POWER SERIES

Work the following on **notebook paper**. Do **not** use your calculator. Show all work.

1. (a) Find a Maclaurin series for $f(x) = \cos x$. Give the first four nonzero terms and the general term.

   (b) Use your answer to (a) to find $\lim\limits_{x\to 0}\frac{\cos x - 1}{x^2}$.

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2. (a) Find a Maclaurin series for $f(x) = \frac{1}{1-2x}$. Give the first four nonzero terms and the general term.

   (b) Use your answer to (a) to find $\lim\limits_{x\to 0}\frac{f(x)-1}{x}$.

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3. (a) Find a Maclaurin series for $f(x) = \sin x$. Give the first four nonzero terms and the general term.

   (b) Use your answer to (a) to approximate the value of $\int_0^1 \frac{\sin t}{t}\,dt$ so that the error in your approximation is less than $\frac{1}{500}$. Justify your answer.

On problems 4 - 5, find a series for the given function. Give the first four nonzero terms and the general term for the series.

4. $f(x) = e^{(x+2)}$ centered at $x = 0$

5. $g(x) = e^{(x+2)}$ centered at $x = -2$

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6. (a) Let $f(x) = \sin(x^2)$. Write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

   (b) Let $g(x) = \cos(x)$. Write the first four nonzero terms of the Taylor series for $\cos(x^3)$ about $x = 0$.

   (c) Let $h(x) = \sin(x^2) + \cos(x)$. Write the first four nonzero terms of the Taylor series for h about $x = 0$.

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7. (a) Let $f(x) = \sin(x^2)$. Write the first four nonzero terms and the general term of the Taylor series for $\sin(x^2)$ about $x = 0$.

   (b) Let $g'(x) = \sin(x^2)$. Given that $g(0) = 1$, write the first five nonzero terms and the general term of the Taylor series for $g(x)$ about $x = 0$.

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8. (1990 BC 5) Let f be the function defined by $f(x) = \frac{1}{x-1}$.

   (a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

   (b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

   (c) Use the series in part (b) to find an approximation for $\ln\frac{3}{2}$ so that the error in your approximation is less than $\frac{1}{20}$. How many terms were needed? Justify your answer.

Answers

Worksheet 2 on Power Series

1. (a) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^n x^{2n}}{(2n)!}+\ldots$ (b) $-\frac{1}{2}$

2. (a) $1+2x+4x^2+8x^3+\ldots+(2x)^n+\ldots$ (b) 2

3. (a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\ldots$

   (b) $\frac{17}{18}$. Since the terms of the series are alternating, decreasing in magnitude, and having a limit of 0 and the approximation is made by using the first two terms, the error will be less than the absolute value of the third term, so $|\text{Error}|<\frac{1}{600}<\frac{1}{500}$.

4. $e^2+e^2x+\frac{e^2x^2}{2!}+\frac{e^2x^3}{3!}+\ldots+\frac{e^2x^n}{n!}+\ldots$

5. $1+(x+2)+\frac{(x+2)^2}{2!}+\frac{(x+2)^3}{3!}+\ldots+\frac{(x+2)^n}{n!}+\ldots$

6. (a) $x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots$ (b) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

   (c) $1+\left(1-\frac{1}{2!}\right)x^2+\frac{x^4}{4!}-\left(\frac{1}{3!}+\frac{1}{6!}\right)x^6+\ldots=1+\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{121x^6}{6!}+\ldots$

7. (a) $x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\ldots+\frac{(-1)^n x^{4n+2}}{(2n+1)!}+\ldots$

   (b) $1+\frac{x^3}{3}-\frac{x^7}{7\cdot 3!}+\frac{x^{11}}{11\cdot 5!}-\frac{x^{15}}{15\cdot 7!}+\ldots+\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}+\ldots$

8. (a) $1-(x-2)+(x-2)^2-(x-2)^3+\ldots+(-1)^n(x-2)^n+\ldots$

   (b) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}-\frac{(x-2)^4}{4}+\ldots+\frac{(-1)^n(x-2)^{n+1}}{n+1}+\ldots$

   (c) $\ln 2\approx\frac{3}{8}$. Two terms are needed. Since the terms of the series are alternating, decreasing in magnitude, and having a limit of 0 and the approximation is made by using the first two terms, the error will be less than the absolute value of the third term, so $|\text{Error}|<\frac{1}{24}<\frac{1}{20}=0.05$.