Area and Volume

1. (*10 points*) Find the area of the region bounded by the given curves.

a. (*5 pts*) $y = \sin x,\ y = \cos x,\ x = \frac{\pi}{6},\ x = \frac{\pi}{3}$

b. (*5 pts*) $x = y^3 - y^2 + 4,\ x = y^2 + 1,\ y = -1,\ y = 2$

2. (*10 points*) Find the area of the region enclosed by the given curves.

a. (*5 pts*) $y = (x-2)^2,\ y = x$

b. (*5 pts*) $x = y^2 - 4y,\ x = 2y$

Area and Volume

Answers

1. (*10 points*) Find the area of the region bounded by the given curves.

a. (*5 pts*) $2\sqrt{2} - 1 - \sqrt{3}$

b. (*5 pts*) $\dfrac{27}{4}$

2. (*10 points*) Find the area of the region enclosed by the given curves.

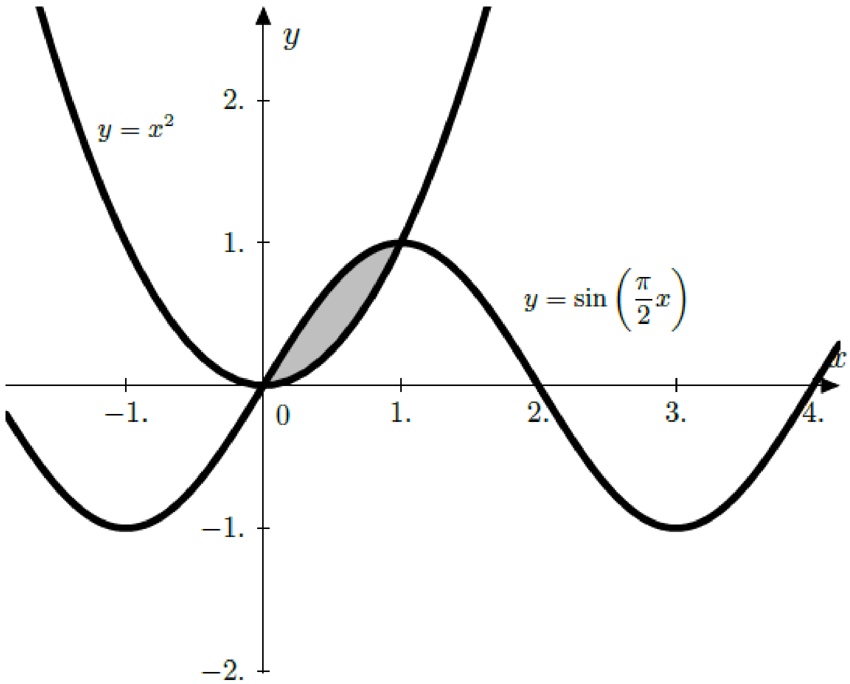
a. (*5 pts*) $\dfrac{9}{2}$

b. (*5 pts*) 36

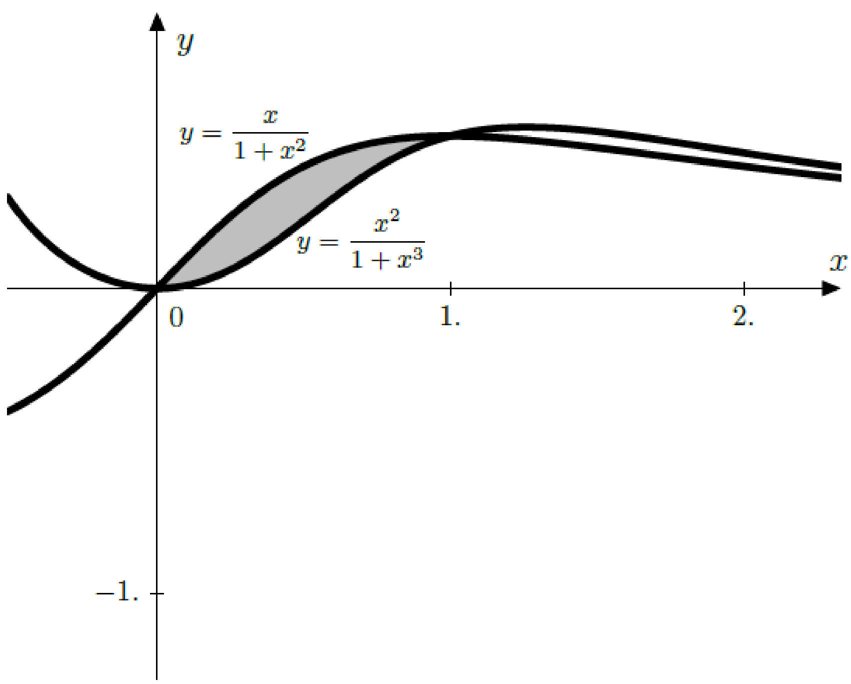
Area and Volume

3. (*15 points*) Find the indicated area.

a. (*5 pts*)



b. (*5 pts*)



Area and Volume

Answers

3. (*15 points*) Find the indicated area.

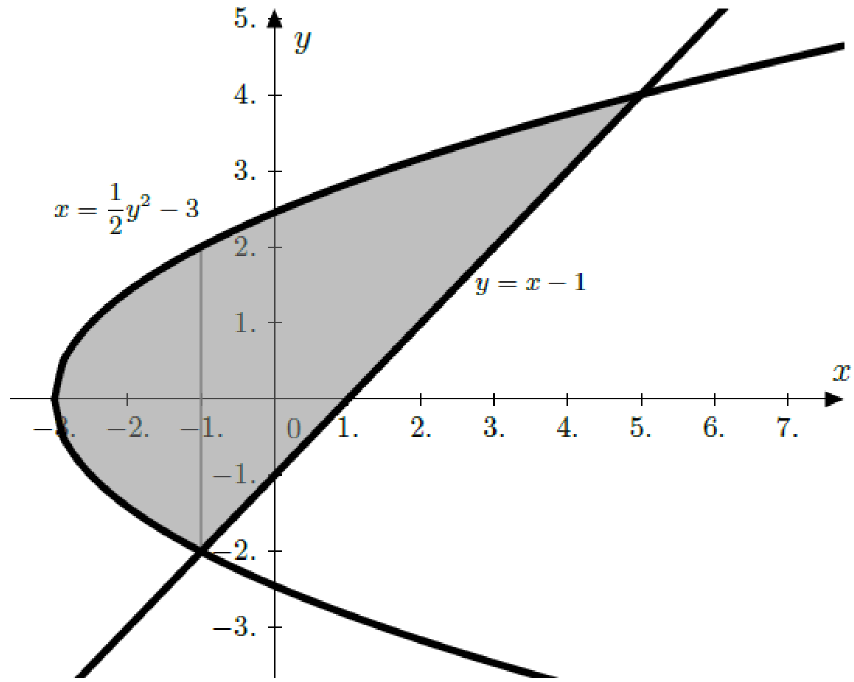
a. (*5 pts*) $\dfrac{2}{\pi} - \dfrac{1}{3}$

c. (*5 pts*) 18

b. (*5 pts*) $\dfrac{\ln 2}{6}$

Area and Volume

c. (*5 pts*)



Area and Volume

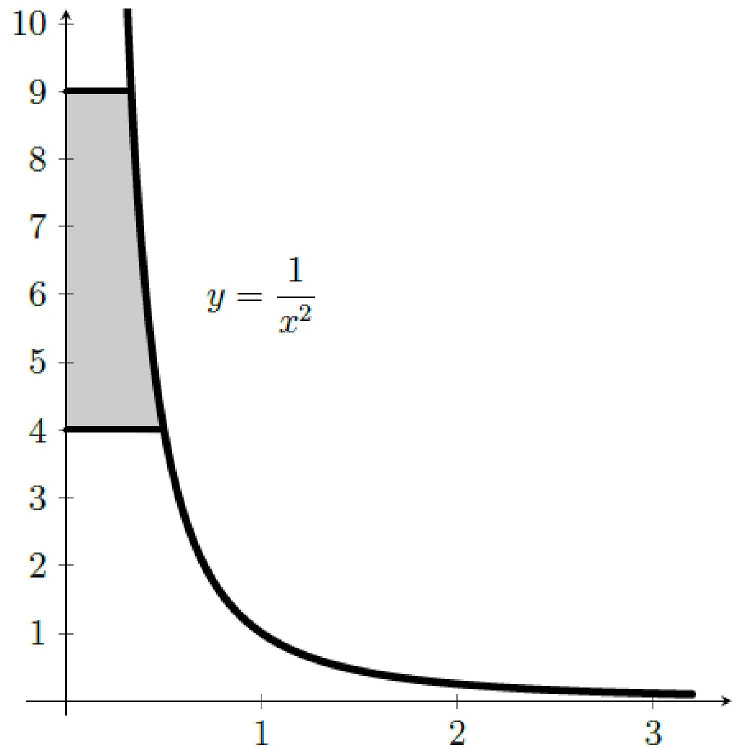
Answers

c. (*5 pts*) 18

Area and Volume

4. (*24 points*) Find the volume of the solid of revolution generated by rotating the given area about the given axis.

a. (*6 pts*) $y = \dfrac{1}{x^2},\ \dfrac{1}{3} \le x \le \dfrac{1}{2}$; about the y-axis



Area and Volume

Answers

4. (*24 points*) Find the volume of the solid of revolution generated by rotating the given area about the given axis.

a. (*6 pts*) $\pi \ln\left(\frac{9}{4}\right)$

b. (*6 pts*) $\frac{3\pi}{10}$

5. (*10 points*) Find the arc length of the graph of the given function over the indicated interval.

a. (*5 pts*) $y = \ln(\sec x);\ [0, \pi/4]$

b. (*5 pts*) $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y;\ 1 \le y \le 2$

Area and Volume

Answers

5. (*10 points*) Find the arc length of the graph of the given function over the indicated interval.

a. (*5 pts*) $\ln(\sqrt{2}+1)$

b. (*5 pts*) $\dfrac{3+2\ln 2}{4}$

Area and Volume

6. (*12 points*) Find the area of the surface generated by rotating the given curve about the indicated axis over the indicated interval.

a. (*6 pts*) $y = \sqrt{9 - x^2}$; $-2 \le x \le 2$; about the x-axis

b. (*6 pts*) $y = \sqrt{\dfrac{x-1}{2}}$; $1 \le y \le 2$; about the x-axis

Area and Volume

Answers

6. (*12 points*) Find the area of the surface generated by rotating the given curve about the indicated axis over the indicated interval.

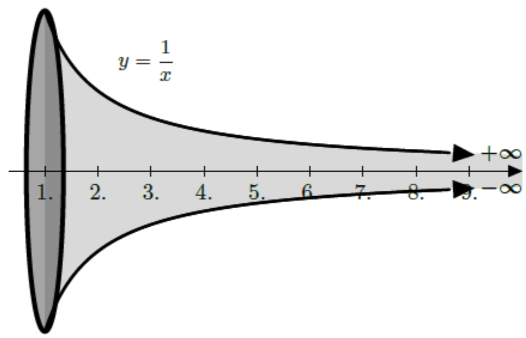
a. (*6 pts*) 24π

b. (*6 pts*) $\frac{\pi}{24}(65\sqrt{65} - 17\sqrt{17})$

Area and Volume

7. (*10 points*) **The Paradox of Gabriel's Horn**

Gabriel's Horn is described as an infinitely long horn, tapering to an infinitesimal end. The horn itself is generated by revolving about the x-axis the unbounded region ($x \geq 1$) between $y = x^{-1}$ and the x-axis. The graph is given below.



Imagine you are told to paint the inside of the horn. Well, it's an infinite horn, so there's no way you can do this, right? Let's consider 2 ways to paint the inside of a canister. One way is to paint the inside surface, stroke by stroke. The other way is to fill the canister with paint and that would coat the surface. Once you dump the excess paint out, you have painted the inside of the canister. Each method is equally valid. To calculate how much paint you would need, the first method involves finding the surface area painted. The second method involves finding the volume of the solid. ***Show that the volume is finite and that the surface area is infinite***. You can (and should) calculate the volume directly. This will give you a numerical answer. You do not need to directly calculate the surface area. Once you have the correct integral set up, you can use the Comparison Theorem to show that the integral is divergent and, hence, the surface area is infinite. Hint: You should use the Comparison Theorem with $y = \frac{1}{x}$.

Area and Volume

Answers

7. (*10 points*) For the volume (disk method):

$$V = \int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx = \pi\left(\lim_{c\to\infty} \int_1^c x^{-2}\, dx\right)$$

$$= \pi\left(\lim_{c\to\infty} \Big[-x^{-1}\Big]_1^c\right)$$

$$= \pi\left[\lim_{c\to\infty}\left(-\frac{1}{c}+\frac{1}{1}\right)\right]$$

$$= \pi\left[\lim_{c\to\infty}\left(1-\frac{1}{c}\right)\right]$$

$$= \pi(1) = \pi$$

For surface area (x as the variable):

Note first that over the interval $[1,\infty)$, we have $x^4+1 > x^4$.

Thus, $\sqrt{x^4+1} > \sqrt{x^4} = x^2$.

All together, $\sqrt{x^4+1} > x^2$.

$$A = \int_1^{\infty} 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx = 2\pi\left(\lim_{c\to\infty}\int_1^c \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx\right)$$

$$= 2\pi\left(\lim_{c\to\infty}\int_1^c \frac{1}{x}\sqrt{\frac{x^4}{x^4}+\frac{1}{x^4}}\, dx\right)$$

Area and Volume

Answers

$$= 2\pi\left(\lim_{c\to\infty}\int_1^c \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx\right)$$

$$= 2\pi\left(\lim_{c\to\infty}\int_1^c \frac{1}{x}\frac{\sqrt{x^4+1}}{x^2}\,dx\right)$$

$$= 2\pi\left(\lim_{c\to\infty}\int_1^c \frac{\sqrt{x^4+1}}{x^3}\,dx\right)$$

$$> 2\pi\left(\lim_{c\to\infty}\int_1^c \frac{x^2}{x^3}\,dx\right) \qquad \textbf{(Here is the inequality)}$$

$$= 2\pi\left(\lim_{c\to\infty}\int_1^c \frac{dx}{x}\right)$$

Here $p = 1$, so we know this integral is divergent. Thus, by comparison, our surface area is divergent as well, thereby illustrating the paradox of Gabriel's Horn.

Area and Volume

8. (*5 points*) A cylindrical tank with a height of 8 m and a radius of 2m is full of water. How much work is required to pump all of the water out of the top of the tank? Use 9.81 $\frac{m}{s^2}$ for acceleration due to gravity and 1000 $\frac{kg}{m^3}$ as the density of water.

Area and Volume

Answers

8. *(5 points)* $1,255,680\pi$ J

Area and Volume

9. (*5 points*) A force of 3 N is needed to stretch a spring from its natural position of 5 cm to a length of 17 cm. How much work is done to stretch the spring from a length of 53 cm to a length of 1 m?

Area and Volume

Answers

9. (*5 points*) 8.40125 J

Area and Volume

10. (*5 points*) A flat isosceles right triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate. The weight-density of water is 62.4 $\frac{lbs}{ft^3}$.

Area and Volume

Answers

10. (*5 points*) 1684.8 lb.

Area and Volume

11. (*5 points*) At 1:00 P.M. on January 15, 1919, a 90-ft-high, 90-ft-diameter cylindrical metal tank in which the Puritan Distilling Company was storing molasses at the corner of Foster and Commercial streets in Boston's North End exploded. In fact, the lowest 1 foot of the tank wall detached. Given that the molasses weighed 100 $\mathrm{lb/ft^3}$, find the force exerted against the bottom 1 ft of the wall.

Area and Volume

Answers

11. (*5 points*) 2,530,553 lb.