Recall the

Theorem on Local Extrema

If $f(c)$ is a local extremum, then either f is not differentiable at c or $f'(c) = 0$.

We will use this to prove

Rolle's Theorem

Let $a < b$. If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) = f(b)$, then there is a c in (a, b) with $f'(c) = 0$. That is, under these hypotheses, f has a horizontal tangent somewhere between a and b.

Rolle's Theorem, like the Theorem on Local Extrema, ends with $f'(c) = 0$. The proof of Rolle's Theorem is a matter of examining cases and applying the Theorem on Local Extrema,

Proof of Rolle's Theorem

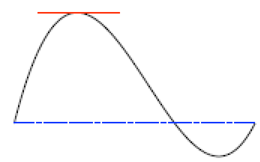
We seek a c in (a, b) with $f'(c) = 0$. That is, we wish to show that f has a horizontal tangent somewhere between a and b. Keep in mind that $f(a) = f(b)$.

Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem says that f has a maximum value $f(M)$ and a minimum value $f(m)$ on the closed interval $[a, b]$. Either $f(M) = f(m)$ or $f(M) \neq f(m)$.
First we suppose the maximum value $f(M) = f(m)$, the minimum value. So all values of f on $[a, b]$ are equal, and f is constant on $[a, b]$. Then $f'(x) = 0$ for all x in (a, b). So one may take c to be anything in (a, b); for example, $c = \frac{a+b}{2}$ would suffice.

Proof of Rolle's Theorem

Now we suppose $f(M) \neq f(m)$. So at least one of $f(M)$ and $f(m)$ is not equal to the value $f(a) = f(b)$.

We first consider the case where the maximum value $f(M) \neq f(a) = f(b)$. So $a \neq M \neq b$. But M is in $[a, b]$ and not at the end points. Thus M is in the open interval (a, b). $f(M) \geq f(x)$ for all x in the closed interval $[a, b]$ which contains the open interval (a, b). So we also have $f(M) \geq f(x)$ for all x in the open interval (a, b). This means that $f(M)$ is a local maximum. Since f is differentiable on (a, b), the Theorem on Local Extrema says $f'(M) = 0$. So we take $c = M$, and we are done with this case.



The case with the minimum value $f(m) \neq f(a) = f(b)$ is similar and left for you to do.

So we are done with the proof of Rolle's Theorem.

joint application of Rolle's Theorem and the Intermediate Value Theorem

We show that $x^5 + 4x = 1$ has exactly one solution. Let $f(x) = x^5 + 4x$. Since f is a polynomial, f is continuous everywhere. $f'(x) = 5x^4 + 4 \geq 0 + 4 = 4 > 0$ for all x. So $f'(x)$ is never 0. So by Rolle's Theorem, no equation of the form $f(x) = C$ can have 2 or more solutions. In particular $x^5 + 4x = 1$ has at most one solution.
$f(0) = 0^5 + 4 \cdot 0 = 0 < 1 < 5 = 1 + 4 = f(1)$. Since f is continuous everywhere, by the Intermediate Value Theorem, $f(x) = 1$ has a solution in the interval $[0, 1]$.
Together these reults say $x^5 + 4x = 1$ has exactly one solution, and it lies in $[0, 1]$.

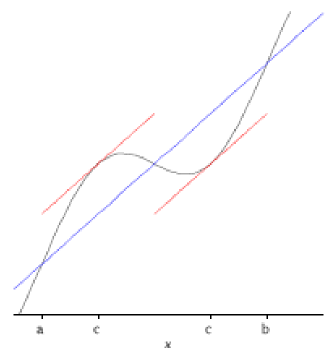
The traditional name of the next theorem is the Mean Value Theorem. A more descriptive name would be Average Slope Theorem.

Mean Value Theorem

Let $a < b$. If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a c in (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is, under appropriate smoothness conditions the slope of the curve at some point between a and b is the same as the slope of the line joining $\langle a, f(a)\rangle$ to $\langle b, f(b)\rangle$. The figure to the right shows two such points, each labeled c.



The Mean Value Theorem generalizes Rolle's Theorem.

Let's look again at the two theorems together.

Rolle's Theorem

Let $a < b$. If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there is a c in (a, b) with $f'(c) = 0$.

Mean Value Theorem

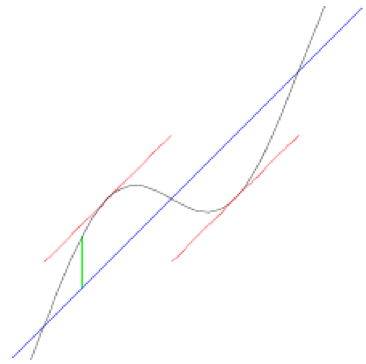
Let $a < b$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The proof of the Mean Value Theorem is accomplished by finding a way to apply Rolle's Theorem. One considers the line joining the points $\langle a, f(a)\rangle$ and $\langle b, f(b)\rangle$. The difference between f and that line is a function that turns out to satisfy the hypotheses of Rolle's Theorem, which then yields the desired result.

Proof of the Mean Value Theorem

Suppose f satisfies the hypotheses of the Mean Value Theorem. We let g be the difference between f and the line joining the points $\langle a, f(a)\rangle$ and $\langle b, f(b)\rangle$. That is, $g(x)$ is the height of the vertical green line in the figure to the right.



The line joining the points $\langle a, f(a)\rangle$ and $\langle b, f(b)\rangle$ has equation

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Proof of the Mean Value Theorem

So

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right].$$

g is the difference of two continuous functions. So g is continuous on $[a, b]$.
g is the difference of two differentiable functions. So g is differentiable on (a, b). Moreover, the derivative of g is the difference between the derivative of f and the derivative (slope) of the line. That is,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Proof of the Mean Value Theorem

Both f and the line go through the points $\langle a, f(a)\rangle$ and $\langle b, f(b)\rangle$. So the difference between them is 0 at a and at b. Indeed,

$$g(a) = f(a) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(a - a)\right] = f(a) - [f(a) + 0] = 0,$$

and

$$\begin{aligned} g(b) &= f(b) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(b - a)\right] \\ &= f(b) - [f(a) + f(b) - f(a)] = 0. \end{aligned}$$

Proof of the Mean Value Theorem

So Rolle's Theorem applies to g. So there is a c in the open interval (a, b) with $g'(c) = 0$. Above we calculated that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Using that we have

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

which is what we needed to prove.

Example

We illustrate The Mean Value Theorem by considering $f(x) = x^3$ on the interval $[1, 3]$.
f is a polynomial and so continuous everywhere. For any x we see that $f'(x) = 3x^2$. So f is continuous on $[1, 3]$ and differentiable on $(1, 3)$. So the Mean Value theorem applies to f and $[1, 3]$.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 1}{2} = 13.$$

$f'(c) = 3c^2$. So we seek a c in $[1, 3]$ with $3c^2 = 13$.

Example

$3c^2 = 13$ iff $c^2 = \frac{13}{3}$ iff $c = \pm\sqrt{\frac{13}{3}}$.
$-\sqrt{\frac{13}{3}}$ is not in the interval $(1, 3)$, but $\sqrt{\frac{13}{3}}$ is a little bigger than $\sqrt{\frac{12}{3}} = \sqrt{4} = 2$. So $\sqrt{\frac{13}{3}}$ is in the interval $(1, 3)$.
So $c = \sqrt{\frac{13}{3}}$ is in the interval $(1, 3)$, and

$$f'(c) = f'\left(\sqrt{\frac{13}{3}}\right) = 13 = \frac{f(3) - f(1)}{3 - 1} = \frac{f(b) - f(a)}{b - a}.$$