Trigonometric Identities

Prove each of the following identities.

1. $\tan x \sin x + \cos x = \sec x$

2. $\dfrac{1}{\tan x} + \tan x = \dfrac{1}{\sin x \cos x}$

3. $\sin x - \sin x \cos^2 x = \sin^3 x$

4. $\dfrac{\cos \alpha}{1 + \sin \alpha} + \dfrac{1 + \sin \alpha}{\cos \alpha} = 2 \sec \alpha$

5. $\dfrac{\cos x}{1 - \sin x} - \dfrac{\cos x}{1 + \sin x} = 2 \tan x$

Trigonometric Identities

Answers

1. $\tan x \sin x + \cos x = \sec x$

   Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\begin{aligned}
\text{LHS} &= \tan x \sin x + \cos x = \frac{\sin x}{\cos x} \cdot \sin x + \cos x = \frac{\sin^2 x}{\cos x} + \cos x = \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} = \text{RHS}
\end{aligned}$$

2. $\dfrac{1}{\tan x} + \tan x = \dfrac{1}{\sin x \cos x}$

   Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\text{LHS} = \frac{1}{\tan x} + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \text{RHS}$$

3. $\sin x - \sin x \cos^2 x = \sin^3 x$

   Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\text{LHS} = \sin x - \sin x \cos^2 x = \sin x \left(1 - \cos^2 x\right) = \sin x \cdot \sin^2 x = \text{RHS}$$

4. $\dfrac{\cos \alpha}{1 + \sin \alpha} + \dfrac{1 + \sin \alpha}{\cos \alpha} = 2 \sec \alpha$

   Solution: We will only use the fact that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\begin{aligned}
\text{LHS} &= \frac{\cos \alpha}{1 + \sin \alpha} + \frac{1 + \sin \alpha}{\cos \alpha} = \frac{\cos^2 \alpha}{(1 + \sin \alpha) \cos \alpha} + \frac{(1 + \sin \alpha)^2}{(1 + \sin \alpha) \cos \alpha} = \frac{\cos^2 \alpha + (1 + \sin \alpha)^2}{(1 + \sin \alpha) \cos \alpha} \\
&= \frac{\cos^2 \alpha + 1 + 2 \sin \alpha + \sin^2 \alpha}{(1 + \sin \alpha) \cos \alpha} = \frac{\cos^2 \alpha + \sin^2 \alpha + 1 + 2 \sin \alpha}{(1 + \sin \alpha) \cos \alpha} = \frac{2 + 2 \sin \alpha}{(1 + \sin \alpha) \cos \alpha} \\
&= \frac{2(1 + \sin \alpha)}{(1 + \sin \alpha) \cos \alpha} = \frac{2}{\cos \alpha} = 2 \cdot \frac{1}{\cos \alpha} = 2 \sec \alpha = \text{RHS}
\end{aligned}$$

5. $\dfrac{\cos x}{1 - \sin x} - \dfrac{\cos x}{1 + \sin x} = 2 \tan x$

   Solution: We will start with the left-hand side. First we bring the fractions to the common denominator. Recall that $\sin^2 x + \cos^2 x = 1$ for all values of x.

$$\begin{aligned}
\text{LHS} &= \frac{\cos x}{1 - \sin x} - \frac{\cos x}{1 + \sin x} = \frac{\cos x (1 + \sin x)}{(1 - \sin x)(1 + \sin x)} - \frac{\cos x (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} \\
&= \frac{\cos x (1 + \sin x) - \cos x (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} = \frac{\cos x + \cos x \sin x - \cos x + \cos x \sin x}{1 - \sin^2 x} = \frac{2 \sin x \cos x}{\cos^2 x} \\
&= \frac{2 \sin x}{\cos x} = 2 \tan x = \text{RHS}
\end{aligned}$$

Prove each of the following identities.

6. $\cos^2 x = \dfrac{\csc x \cos x}{\tan x + \cot x}$

7. $\dfrac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1$

8. $\dfrac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$

9. $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

6. $\cos^2 x = \dfrac{\csc x \cos x}{\tan x + \cot x}$

Solution: We will start with the right-hand side. We will re-write everything in terms of $\sin x$ and $\cos x$ and simplify. We will again run into the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$.

$$\text{RHS} = \frac{\csc x \cos x}{\tan x + \cot x} = \frac{\dfrac{1}{\sin x} \cdot \cos x}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}} = \frac{\dfrac{1}{\sin x} \cdot \dfrac{\cos x}{1}}{\dfrac{\sin^2 x}{\sin x \cos x} + \dfrac{\cos^2 x}{\sin x \cos x}} = \frac{\dfrac{\cos x}{\sin x}}{\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}} = \frac{\dfrac{\cos x}{\sin x}}{\dfrac{1}{\sin x \cos x}}$$

$$= \frac{\cos x}{\sin x} \cdot \frac{\cos x \sin x}{1} = \frac{\cos^2 x}{1} = \cos^2 x = \text{LHS}$$

7. $\dfrac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1$

Solution: We can factor the numerator via the difference of squares theorem.

$$\text{LHS} = \frac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = \frac{\left(\sin^2 x\right)^2 - \left(\cos^2 x\right)^2}{\sin^2 x - \cos^2 x} = \frac{\left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)}{\sin^2 x - \cos^2 x}$$

$$= \sin^2 x + \cos^2 x = 1 = \text{RHS}$$

8. $\dfrac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$

Solution:

$$\text{LHS} = \frac{\tan^2 x}{\tan^2 x + 1} = \frac{\left(\dfrac{\sin x}{\cos x}\right)^2}{\left(\dfrac{\sin x}{\cos x}\right)^2 + 1} = \frac{\dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{\sin^2 x}{\cos^2 x} + 1} = \frac{\dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x}}$$

$$= \frac{\dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{\sin^2 x + \cos^2 x}{\cos^2 x}} = \frac{\dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{1}{\cos^2 x}} = \frac{\sin^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{1} = \sin^2 x = \text{RHS}$$

9. $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

Solution:

$$\text{LHS} = \frac{1 - \sin x}{\cos x} = \frac{1 - \sin x}{\cos x} \cdot 1 = \frac{1 - \sin x}{\cos x} \cdot \frac{1 + \sin x}{1 + \sin x} = \frac{(1 - \sin x)(1 + \sin x)}{\cos x (1 + \sin x)} = \frac{1 - \sin^2 x}{\cos x (1 + \sin x)}$$

$$= \frac{\cos^2 x}{\cos x (1 + \sin x)} = \frac{\cos x}{1 + \sin x} = \text{RHS}$$

Prove each of the following identities.

10. $1 - 2\cos^2 x = \dfrac{\tan^2 x - 1}{\tan^2 x + 1}$

11. $\tan^2 \theta = \csc^2 \theta \tan^2 \theta - 1$

12. $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

13. $\dfrac{\csc \beta}{\sin \beta} - \dfrac{\cot \beta}{\tan \beta} = 1$

14. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

10. $1 - 2\cos^2 x = \dfrac{\tan^2 x - 1}{\tan^2 x + 1}$

Solution:

$$\begin{aligned}
\text{RHS} &= \frac{\tan^2 x - 1}{\tan^2 x + 1} = \frac{\dfrac{\sin^2 x}{\cos^2 x} - 1}{\dfrac{\sin^2 x}{\cos^2 x} + 1} = \frac{\dfrac{\sin^2 x}{\cos^2 x} - \dfrac{\cos^2 x}{\cos^2 x}}{\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x}} = \frac{\dfrac{\sin^2 x - \cos^2 x}{\cos^2 x}}{\dfrac{\sin^2 x + \cos^2 x}{\cos^2 x}} \\
&= \frac{\sin^2 x - \cos^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{1} = \sin^2 x - \cos^2 x \\
&= \left(1 - \cos^2 x\right) - \cos^2 x = 1 - 2\cos^2 x = \text{LHS}
\end{aligned}$$

11. $\tan^2 \theta = \csc^2 \theta \tan^2 \theta - 1$

$$\begin{aligned}
\text{RHS} &= \csc^2 \theta \tan^2 \theta - 1 = \frac{1}{\sin^2 \theta} \cdot \left(\frac{\sin \theta}{\cos \theta}\right)^2 - 1 = \frac{1}{\sin^2 \theta} \cdot \frac{\sin^2 \theta}{\cos^2 \theta} - 1 = \frac{1}{\cos^2 \theta} - 1 \\
&= \frac{1}{\cos^2 \theta} - \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1 - \cos^2 \theta}{\cos^2 \theta} = \frac{\sin^2 \theta}{\cos^2 \theta} = \left(\frac{\sin \theta}{\cos \theta}\right)^2 = \tan^2 \theta = \text{LHS}
\end{aligned}$$

12. $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

Solution:

$$\begin{aligned}
\text{RHS} &= \frac{\cos x}{1 - \sin x} = \frac{\cos x}{1 - \sin x} \cdot 1 = \frac{\cos x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} = \frac{\cos x \left(1 + \sin x\right)}{\left(1 - \sin x\right)\left(1 + \sin x\right)} \\
&= \frac{\cos x \left(1 + \sin x\right)}{1 - \sin^2 x} = \frac{\cos x \left(1 + \sin x\right)}{\cos^2 x} = \frac{1 + \sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \text{LHS}
\end{aligned}$$

13. $\dfrac{\csc \beta}{\sin \beta} - \dfrac{\cot \beta}{\tan \beta} = 1$

Solution: We will start with the left-hand side. We will re-write everything in terms of $\sin \beta$ a $\cos \beta$ and simplify. We will again run into the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$ for all ang x.

$$\begin{aligned}
\text{LHS} &= \frac{\csc \beta}{\sin \beta} - \frac{\cot \beta}{\tan \beta} = \frac{\dfrac{1}{\sin \beta}}{\sin \beta} - \frac{\dfrac{\cos \beta}{\sin \beta}}{\dfrac{\sin \beta}{\cos \beta}} = \frac{1}{\sin \beta} \cdot \frac{1}{\sin \beta} - \frac{\cos \beta}{\sin \beta} \cdot \frac{\cos \beta}{\sin \beta} = \frac{1}{\sin^2 \beta} - \frac{\cos^2 \beta}{\sin^2 \beta} \\
&= \frac{1 - \cos^2 \beta}{\sin^2 \beta} = \frac{\left(\sin^2 \beta + \cos^2 \beta\right) - \cos^2 \beta}{\sin^2 \beta} = \frac{\sin^2 \beta}{\sin^2 \beta} = 1 = \text{RHS}
\end{aligned}$$

14. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

Solution:

$$\begin{aligned}
\text{LHS} &= \sin^4 x - \cos^4 x = \left(\sin^2 x\right)^2 - \left(\cos^2 x\right)^2 = \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right) \\
&= 1 \cdot \left(\sin^2 x - \cos^2 x\right) = \left(1 - \cos^2 x\right) - \cos^2 x = 1 - 2\cos^2 x = \text{RHS}
\end{aligned}$$

Prove each of the following identities.

15. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$

16. $\dfrac{\sin^2 x + 4\sin x + 3}{\cos^2 x} = \dfrac{3 + \sin x}{1 - \sin x}$

17. $\dfrac{\cos x}{1 - \sin x} - \tan x = \sec x$

18. $\tan^2 x + 1 + \tan x \sec x = \dfrac{1 + \sin x}{\cos^2 x}$

Trigonometric Identities

Answers

15. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$

    Solution:

    $$\begin{aligned}
    \text{LHS} &= (\sin x - \cos x)^2 + (\sin x + \cos x)^2 \\
    &= \left(\sin^2 x + \cos^2 x - 2\sin x \cos x\right) + \left(\sin^2 x + \cos^2 x + 2\sin x \cos x\right) = 2\sin^2 x + 2\cos^2 x \\
    &= 2\left(\sin^2 x + \cos^2 x\right) = 2 \cdot 1 = 2 = \text{RHS}
    \end{aligned}$$

16. $\dfrac{\sin^2 x + 4\sin x + 3}{\cos^2 x} = \dfrac{3 + \sin x}{1 - \sin x}$

    Solution:

    $$\text{LHS} = \frac{\sin^2 x + 4\sin x + 3}{\cos^2 x} = \frac{(\sin x + 1)(\sin x + 3)}{1 - \sin^2 x} = \frac{(\sin x + 1)(\sin x + 3)}{(1 + \sin x)(1 - \sin x)} = \frac{\sin x + 3}{1 - \sin x} = \text{RHS}$$

17. $\dfrac{\cos x}{1 - \sin x} - \tan x = \sec x$

    Solution:

    $$\begin{aligned}
    \text{LHS} &= \frac{\cos x}{1 - \sin x} - \tan x = \frac{\cos x}{1 - \sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin x (1 - \sin x)}{\cos x (1 - \sin x)} = \frac{\cos^2 x - \sin x + \sin^2 x}{\cos x (1 - \sin x)} \\
    &= \frac{\left(\cos^2 x + \sin^2 x\right) - \sin x}{\cos x (1 - \sin x)} = \frac{1 - \sin x}{\cos x (1 - \sin x)} = \frac{1}{\cos x} = \text{RHS}
    \end{aligned}$$

18. $\tan^2 x + 1 + \tan x \sec x = \dfrac{1 + \sin x}{\cos^2 x}$

    Solution:

    $$\begin{aligned}
    \text{LHS} &= \tan^2 x + 1 + \tan x \sec x = \frac{\sin^2 x}{\cos^2 x} + 1 + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \\
    &= \frac{\sin^2 x + \cos^2 x + \sin x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x} = \text{RHS}
    \end{aligned}$$