Addition and Subtraction Identities

In this section, we begin expanding our repertoire of trigonometric identities.

Identities

The sum and difference identities

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Example 1

Find the exact value of $\cos(75°)$.

Since $75° = 30° + 45°$, we can evaluate $\cos(75°)$ as

$\cos(75°) = \cos(30° + 45°)$	Apply the cosine sum of angles identity
$= \cos(30°)\cos(45°) - \sin(30°)\sin(45°)$	Evaluate
$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$	Simply
$= \frac{\sqrt{6} - \sqrt{2}}{4}$	

Try it Now

2. Find the exact value of $\sin\left(\frac{\pi}{12}\right)$.

Example 2

Rewrite $\sin\left(x-\frac{\pi}{4}\right)$ in terms of $\sin(x)$ and $\cos(x)$.

$\sin\left(x-\frac{\pi}{4}\right)$ — Use the difference of angles identity for sine

$=\sin(x)\cos\left(\frac{\pi}{4}\right)-\cos(x)\sin\left(\frac{\pi}{4}\right)$ — Evaluate the cosine and sine and rearrange

$=\frac{\sqrt{2}}{2}\sin(x)-\frac{\sqrt{2}}{2}\cos(x)$

Example 3

Prove $\dfrac{\sin(a+b)}{\sin(a-b)} = \dfrac{\tan(a)+\tan(b)}{\tan(a)-\tan(b)}$.

As with any identity, we need to first decide which side to begin with. Since the left side involves sum and difference of angles, we might start there

$$\frac{\sin(a+b)}{\sin(a-b)}$$

Apply the sum and difference of angle identities

$$= \frac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\sin(a)\cos(b)-\cos(a)\sin(b)}$$

Since it is not immediately obvious how to proceed, we might start on the other side, and see if the path is more apparent.

$$\frac{\tan(a)+\tan(b)}{\tan(a)-\tan(b)}$$

Rewriting the tangents using the tangent identity

$$= \frac{\dfrac{\sin(a)}{\cos(a)}+\dfrac{\sin(b)}{\cos(b)}}{\dfrac{\sin(a)}{\cos(a)}-\dfrac{\sin(b)}{\cos(b)}}$$

Multiplying the top and bottom by $\cos(a)\cos(b)$

$$= \frac{\left(\dfrac{\sin(a)}{\cos(a)}+\dfrac{\sin(b)}{\cos(b)}\right)\cos(a)\cos(b)}{\left(\dfrac{\sin(a)}{\cos(a)}-\dfrac{\sin(b)}{\cos(b)}\right)\cos(a)\cos(b)}$$

Distributing and simplifying

$$= \frac{\sin(a)\cos(a)+\sin(b)\cos(b)}{\sin(a)\cos(a)-\sin(b)\cos(b)}$$

From above, we recognize this

$$= \frac{\sin(a+b)}{\sin(a-b)}$$

Establishing the identity

Example 4

Solve $\sin(x)\sin(2x)+\cos(x)\cos(2x)=\frac{\sqrt{3}}{2}$.

By recognizing the left side of the equation as the result of the difference of angles identity for cosine, we can simplify the equation

$\sin(x)\sin(2x)+\cos(x)\cos(2x)=\frac{\sqrt{3}}{2}$ Apply the difference of angles identity

$\cos(x-2x)=\frac{\sqrt{3}}{2}$

$\cos(-x)=\frac{\sqrt{3}}{2}$ Use the negative angle identity

$\cos(x)=\frac{\sqrt{3}}{2}$

Since this is a special cosine value we recognize from the unit circle, we can quickly write the answers:

$x=\frac{\pi}{6}+2\pi k$

$x=\frac{11\pi}{6}+2\pi k$

, where k is an integer