Trigonometric Addition and Subtraction

EXAMPLE 1 ■ Using the Addition and Subtraction Formulas

Find the exact value of each expression.

(a) $\cos 75°$ **(b)** $\cos \dfrac{\pi}{12}$

SOLUTION

(a) Notice that $75° = 45° + 30°$. Since we know the exact values of sine and cosine at 45° and 30°, we use the Addition Formula for Cosine to get

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

(b) Since $\dfrac{\pi}{12} = \dfrac{\pi}{4} - \dfrac{\pi}{6}$, the Subtraction Formula for Cosine gives

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

Trigonometric Addition and Subtraction

EXAMPLE 2 ■ Using the Addition Formula for Sine

Find the exact value of the expression $\sin 20° \cos 40° + \cos 20° \sin 40°$.

SOLUTION We recognize the expression as the right-hand side of the Addition Formula for Sine with $s = 20°$ and $t = 40°$. So we have

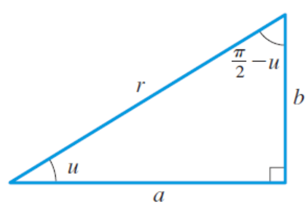
$$\sin 20° \cos 40° + \cos 20° \sin 40° = \sin(20° + 40°) = \sin 60° = \frac{\sqrt{3}}{2}$$

EXAMPLE 3 ■ Proving a Cofunction Identity

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - u\right) = \sin u$.

SOLUTION By the Subtraction Formula for Cosine we have

$$\cos\left(\frac{\pi}{2} - u\right) = \cos\frac{\pi}{2}\cos u + \sin\frac{\pi}{2}\sin u$$
$$= 0 \cdot \cos u + 1 \cdot \sin u = \sin u$$



$\cos\left(\frac{\pi}{2} - u\right) = \frac{b}{r} = \sin u$

For acute angles, the cofunction identity in Example 3, as well as the other cofunction identities, can also be derived from the figure in the margin.

EXAMPLE 4 ■ Proving an Identity

Verify the identity $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(\frac{\pi}{4} + x\right)$.

SOLUTION Starting with the right-hand side and using the Addition Formula for Tangent, we get

$$\text{RHS} = \tan\left(\frac{\pi}{4} + x\right) = \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x}$$
$$= \frac{1 + \tan x}{1 - \tan x} = \text{LHS}$$

EXAMPLE 5 ■ An Identity from Calculus

If $f(x) = \sin x$, show that

$$\frac{f(x+h)-f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

SOLUTION

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$ Definition of f

$$= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$ Addition Formula for Sine

$$= \frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h}$$ Factor

$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$ Separate the fraction

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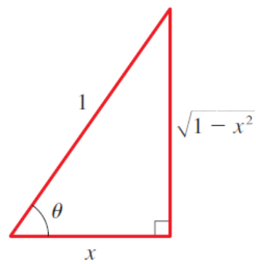
Evaluating Expressions Involving Inverse Trigonometric Functions

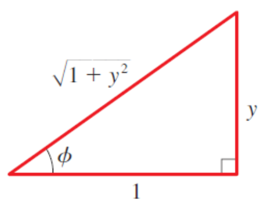
Expressions involving trigonometric functions and their inverses arise in calculus. In the next examples we illustrate how to evaluate such expressions.

EXAMPLE 6 ■ Simplifying an Expression Involving Inverse Trigonometric Functions

Write $\sin(\cos^{-1} x + \tan^{-1} y)$ as an algebraic expression in x and y, where $-1 \le x \le 1$ and y is any real number.



$\cos\theta = x$



$\tan\phi = y$

FIGURE 2

SOLUTION Let $\theta = \cos^{-1} x$ and $\phi = \tan^{-1} y$. Using the methods of Section 6.4, we sketch triangles with angles θ and ϕ such that $\cos\theta = x$ and $\tan\phi = y$ (see Figure 2). From the triangles we have

$$\sin\theta = \sqrt{1 - x^2} \qquad \cos\phi = \frac{1}{\sqrt{1 + y^2}} \qquad \sin\phi = \frac{y}{\sqrt{1 + y^2}}$$

From the Addition Formula for Sine we have

$$\begin{aligned}
\sin(\cos^{-1} x + \tan^{-1} y) &= \sin(\theta + \phi) \\
&= \sin\theta\cos\phi + \cos\theta\sin\phi && \text{Addition Formula for Sine} \\
&= \sqrt{1 - x^2}\,\frac{1}{\sqrt{1 + y^2}} + x\,\frac{y}{\sqrt{1 + y^2}} && \text{From triangles} \\
&= \frac{1}{\sqrt{1 + y^2}}\left(\sqrt{1 - x^2} + xy\right) && \text{Factor } \frac{1}{\sqrt{1 + y^2}}
\end{aligned}$$

Trigonometric Addition and Subtraction

EXAMPLE 7 ■ Evaluating an Expression Involving Trigonometric Functions

Evaluate $\sin(\theta + \phi)$, where $\sin\theta = \frac{12}{13}$ with θ in Quadrant II and $\tan\phi = \frac{3}{4}$ with ϕ in Quadrant III.

SOLUTION We first sketch the angles θ and ϕ in standard position with terminal sides in the appropriate quadrants as in Figure 3. Since $\sin\theta = y/r = \frac{12}{13}$, we can label a side

and the hypotenuse in the triangle in Figure 3(a). To find the remaining side, we use the Pythagorean Theorem.

$$x^2 + y^2 = r^2 \qquad \text{Pythagorean Theorem}$$

$$x^2 + 12^2 = 13^2 \qquad y = 12, \quad r = 13$$

$$x^2 = 25 \qquad \text{Solve for } x^2$$

$$x = -5 \qquad \text{Because } x < 0$$

Similarly, since $\tan\phi = y/x = \frac{3}{4}$, we can label two sides of the triangle in Figure 3(b) and then use the Pythagorean Theorem to find the hypotenuse.

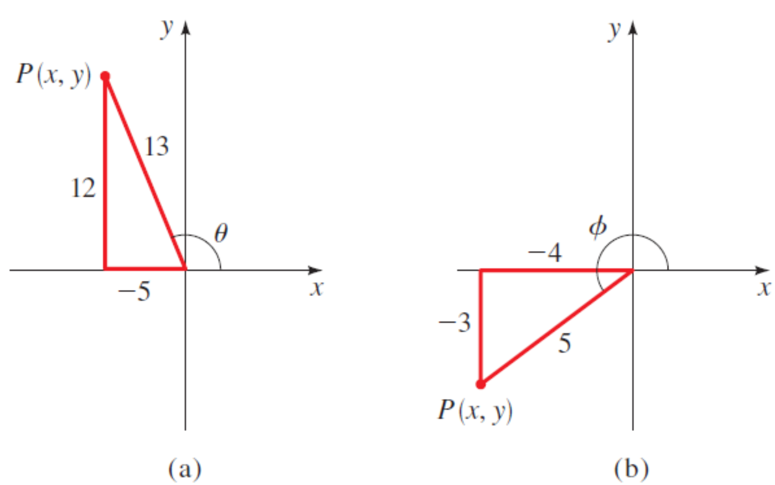


FIGURE 3

Now, to find $\sin(\theta + \phi)$, we use the Addition Formula for Sine and the triangles in Figure 3.

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \qquad \text{Addition Formula}$$

$$= \left(\tfrac{12}{13}\right)\left(-\tfrac{4}{5}\right) + \left(-\tfrac{5}{13}\right)\left(-\tfrac{3}{5}\right) \qquad \text{From triangles}$$

$$= -\tfrac{33}{65} \qquad \text{Calculate}$$