Logarithmic Functions

In this section we introduce logarithmic functions. Notice that every exponential function $f(x) = a^x$, with $a > 0$ and a ≠ 1, is a one-to-one function by the Horizontal Line Test and therefore has an inverse function. The inverse function of the exponential function with base a is called the *logarithmic function with base a* and is denoted by $log_a x$. Recall that f^{-1} is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

This leads to the following definition of the logarithmic function.

Definition of the Logarithmic Function:

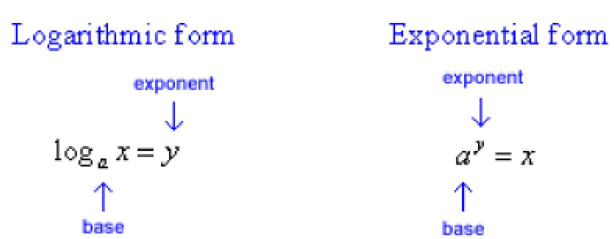
Let a be a positive number with a ≠ 0. The **logarithmic function with base *a***, denoted by **$\log_a$**, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

In other words, this says that

$log_a\ x$ is the exponent to which the base a must be raised to give x.

The form $log_a\ x = y$ is called the **logarithmic form**, and the form $a^y = x$ is called the **exponential form**. Notice that in both forms the base is the same:



Example 1: Express each equation in exponential form.

(a) $\log_7 49 = 2$
(b) $\log_{16} 4 = \frac{1}{2}$

Solution:

From the definition of the logarithmic function we know

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

This implies

(a) $\log_7 49 = 2 \quad \Leftrightarrow \quad 7^2 = 49$

(b) $\log_{16} 4 = \frac{1}{2} \quad \Leftrightarrow \quad 16^{\frac{1}{2}} = 4$

Example 2: Express each equation in logarithmic form.

(a) $3^4 = 81$
(b) $6^{-1} = \frac{1}{6}$

Solution:

From the definition of the logarithmic function we know

$$a^y = x \quad \Leftrightarrow \quad \log_a x = y$$

This implies

(a) $3^4 = 81 \quad \Leftrightarrow \quad \log_3 81 = 4$

(b) $6^{-1} = \frac{1}{6} \quad \Leftrightarrow \quad \log_6 \frac{1}{6} = -1$

Graphs of Logarithmic Functions:

Since the logarithmic function $f(x) = log_a\ x$ is the inverse of the exponential function $f(x) = a^x$, the graphs of these two functions are reflections of each other through the line $y = x$.

Also, since the exponential function with a ≠ 0 has domain ℝ and range (0, ∞), we conclude its inverse, the logarithmic function, has domain (0, ∞) and range ℝ. Finally, since $f(x) = a^x$ has a horizontal asymptote at $y = 0$, $f(x) = log_a\ x$ has a vertical asymptote at $x = 0$.

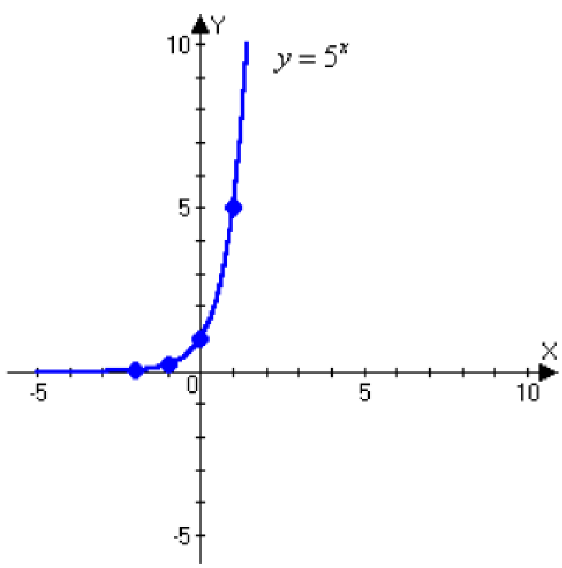
Example 3: Draw the graph of $y = 5^x$, then use it to draw the graph of $y = log_5\ x$.

Solution:

Step 1: To graph $y = 5^x$, start by choosing some values of x and finding the corresponding y-values.

x	y
-2	$\frac{1}{25}$
-1	$\frac{1}{5}$
0	1
1	5
2	25

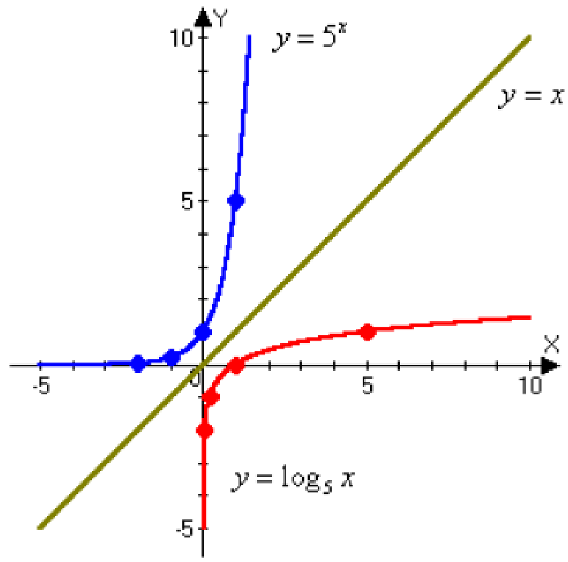
Step 2: Plot the points found in the previous step for $y = 5^x$ and draw a smooth curve connecting them.



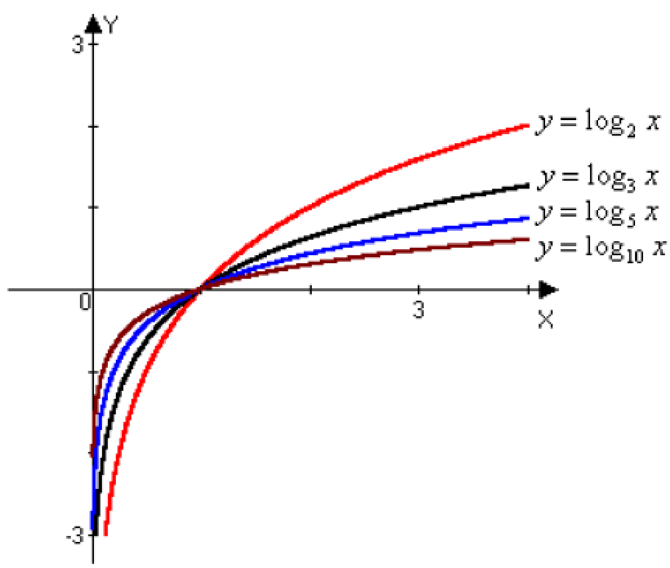
Step 3: To find the graph of $y = log_5\ x$, all we need to do is reflect the graph of $y = 5^x$ over the line $y = x$, because they are inverses.

Another way we can find the graph of $y = log_5\ x$ is to take the chart we found in Step 1 for $y = 5^x$, and switch the x and y values. Then we plot the new points and draw a smooth curve connecting them.

x	y
$\frac{1}{25}$	-2
$\frac{1}{5}$	-1
1	0
5	1
25	2



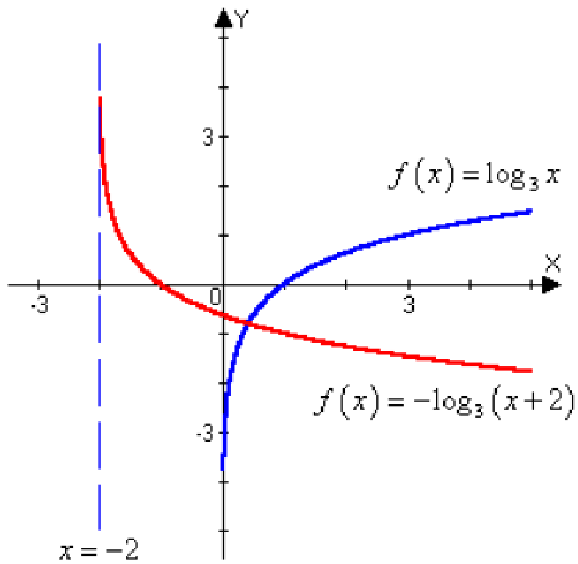
The figure below shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10.



Example 4: Graph the function $f(x) = -log_3(x + 2)$, not by plotting points, but by starting from the graphs in the above figure. State the domain, range, and asymptote.

Solution:

Step 1: To obtain the graph of $f(x) = -log_3(x + 2)$, we start with the graph of $f(x) = log_3 x$, reflect it across the x-axis and shift it to the left 2 units.



Step 2: Notice that while the vertical asymptote is not actually part of the graph, it also shifts left 2 units, and so the vertical asymptote of $f(x) = -log_3(x + 2)$ is the line $x = -2$. Looking at the graph, we see that the domain of f is $(-2, \infty)$, and the range is $\mathbb{R}$.

Properties of Logarithms:

	Property	**Reason**
1.	$\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2.	$\log_a a = 1$	We must raise a to the power 1 to get a.
3.	$\log_a a^x = x$	We must raise a to the power x to get a^x.
4.	$a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Common Logarithms:

Frequently one will see the logarithmic function written without a specified base, $y = \log x$. This is known as the common logarithm, and it is the logarithm with base 10.

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

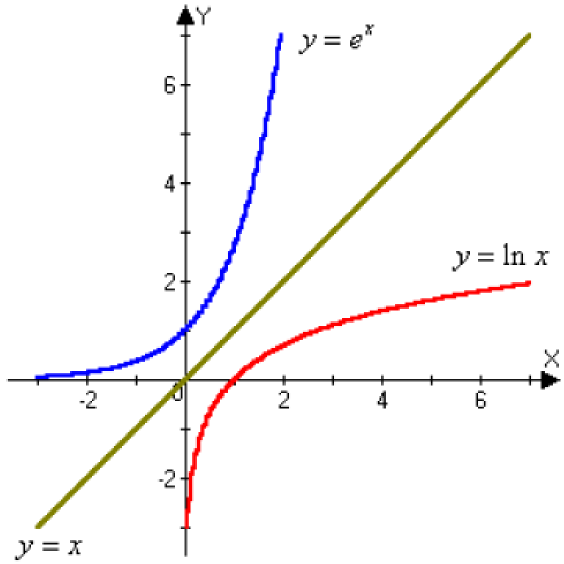
Natural Logarithms:

Of all possible bases a for logarithms, it turns out the most convenient choice for the purposes of calculus is the number e.

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

The natural logarithmic function $y = \ln x$ is the inverse function of the exponential function $y = e^x$. Both functions are graphed below.



By the definition of inverse functions we have

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

The same important properties of logarithms that were listed above also apply to natural logarithms.

Properties of Natural Logarithms:

	Property	Reason
1.	$\ln 1 = 0$	We must raise e to the power 0 to get 1.
2.	$\ln e = 1$	We must raise e to the power 1 to get e.
3.	$\ln e^x = x$	We must raise e to the power x to get e^x.
4.	$e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Example 5: Evaluate the expressions.

(a) $\log_7 1$
(b) $\log_3 3$
(c) $\ln e^{12}$
(d) $10^{\log \pi}$

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(b) $\log_3 3$
(c) $\ln e^{12}$
(d) $10^{\log \pi}$

Solution (a):

The first property of logarithms says $\log_a 1 = 0$. Thus,

$$\log_7 1 = 0$$

Solution (b):

The second property of logarithms says $\log_a a = 1$. Thus,

$$\log_3 3 = 1.$$

Solution (c):

The third property of natural logarithms says $\ln e^x = x$. Thus,

$$\ln e^{12} = 12.$$

Solution (d):

Step 1: First note that $\log \pi = \log_{10} \pi$. So

$$10^{\log \pi} = 10^{\log_{10} \pi}$$

Step 2: The fourth property of logarithms says $a^{\log_a x} = x$. Thus

$$10^{\log_{10} \pi} = \pi.$$

Example 6: Use the definition of the logarithmic function to find x.

(a) $3 = \log_2 x$
(b) $-4 = \log_3 x$
(c) $4 = \log_x 625$
(d) $-2 = \log_x 100$

Solution (a):

Step 1: By the definition of the logarithm, we can rewrite the expression in exponential form.

$$3 = \log_2 x \quad \Leftrightarrow \quad 2^3 = x$$

Step 2: Now we can solve for x.

$$x = 2^3$$
$$x = 8$$

Solution (b):

Step 1: Rewrite the expression in exponential form using the definition of the logarithmic function.

$$-4 = \log_3 x \quad \Leftrightarrow \quad 3^{-4} = x$$

Step 2: Solve for x.

$$x = 3^{-4}$$
$$x = \tfrac{1}{3^4}$$
$$x = \tfrac{1}{81}$$

Solution (c):

Step 1: Rewrite the expression in exponential form using the definition of the logarithmic function.

$$4 = \log_x 625 \quad \Leftrightarrow \quad x^4 = 625$$

Step 2: Solve for x.

$$x^4 = 625 \qquad \text{take the fourth root of both sides}$$
$$x = \pm\sqrt[4]{625}$$
$$x = \pm 5$$

Recall that a logarithm cannot have a negative base. So, we discard the extraneous solution $x = -5$, and therefore $x = 5$ is the only solution to the expression $4 = \log_x 625$.

Solution (d):

Step 1: Rewrite the expression in exponential form using the definition of the logarithmic function.

$$-2 = \log_x 100 \quad \Leftrightarrow \quad x^{-2} = 100$$

Step 2: Solve for x.

$$x^{-2} = 100$$

$$\tfrac{1}{x^2} = 100 \qquad \text{multiply both sides by } x^2$$

$$1 = 100x^2 \qquad \text{divide both sides by } 100$$

$$\tfrac{1}{100} = x^2 \qquad \text{take the square root of both sides}$$

$$x = \pm\sqrt{\tfrac{1}{100}}$$

$$x = \pm\tfrac{1}{10}$$

Again we note that a logarithm cannot have a negative base. So, we discard the extraneous solution $x = -\frac{1}{10}$, and therefore $x = \frac{1}{10}$ is the only solution to the expression $-2 = \log_x 100$.