Definition of a Rational Expression

An expression is a **rational expression** if it can be written in the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Steps to Find the Domain of a Rational Expression

1. Set the denominator equal to zero and solve the resulting equation.
2. The domain is the set of all real numbers *excluding* the values found in step 1.

TIP: The domain of a rational expression excludes values for which the denominator is zero. The values for which the numerator is zero *do not affect* the domain of the expression.

Fundamental Principle of Rational Expressions

Let p, q, and r represent polynomials. Then

$$\frac{pr}{qr} = \frac{p}{q} \qquad \text{for } q \neq 0 \text{ and } r \neq 0$$

Avoiding Mistakes:

The domain of a rational expression is always determined *before* simplifying the expression.

TIP: $t^3 + 8$ is a sum of cubes.

Recall: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$$t^3 + 8 = (t + 2)(t^2 - 2t + 4)$$

Avoiding Mistakes: The fundamental principle of rational expressions indicates that common *factors* in the numerator and denominator may be simplified.

$$\frac{pr}{qr} = \frac{p}{q} \cdot \frac{r}{r} = \frac{p}{q} \cdot (1) = \frac{p}{q}$$

Because this property is based on the identity property of multiplication, reducing applies only to factors (remember that factors are multiplied). Therefore, terms that are added or subtracted cannot be reduced. For example:

$$\frac{3x}{3y} = \frac{\overset{1}{\cancel{3}}x}{\cancel{3}y} = (1) \cdot \frac{x}{y} = \frac{x}{y}$$

Reduce common factor.

However, $\dfrac{x+3}{y+3}$ cannot be simplified.

cannot reduce common terms

TIP: The factor of -1 may be applied in front of the rational expression, or it may be applied to the numerator or to the denominator. Therefore, the final answer may be written in several forms.

$$-\frac{1}{x+5} \quad \text{or} \quad \frac{-1}{x+5} \quad \text{or} \quad \frac{1}{-(x+5)}$$

Definition of a Rational Function

A function f is a **rational function** if it can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions and $q(x) \neq 0$.

Multiplication of Rational Expressions

Let p, q, r, and s represent polynomials, such that $q \neq 0$ and $s \neq 0$. Then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

Multiplying Rational Expressions

1. Factor the numerators and denominators of all rational expressions.
2. Simplify the ratios of common factors to 1.
3. Multiply the remaining factors in the numerator, and multiply the remaining factors in the denominator.

Avoiding Mistakes:

If all factors in the numerator simplify to 1, do not forget to write the factor of 1 in the numerator.

Division of Rational Expressions

Recall that to divide fractions, multiply the first fraction by the reciprocal of the second fraction.

Divide: $\frac{15}{14} \div \frac{10}{49} \xrightarrow[\text{of the second fraction.}]{\text{Multiply by the reciprocal}} \frac{15}{14} \cdot \frac{49}{10} = \frac{3 \cdot \overset{1}{\cancel{5}}}{2 \cdot \cancel{7}} \cdot \frac{\overset{1}{\cancel{7}} \cdot 7}{2 \cdot \cancel{5}} = \frac{21}{4}$

Division of Rational Expressions

Let p, q, r, and s represent polynomials, such that $q \neq 0$, $r \neq 0$, $s \neq 0$. Then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r} = \frac{ps}{qr}$$

Addition and Subtraction of Rational Expressions

Let p, q, and r represent polynomials where $q \neq 0$. Then

1. $\dfrac{p}{q} + \dfrac{r}{q} = \dfrac{p + r}{q}$ **2.** $\dfrac{p}{q} - \dfrac{r}{q} = \dfrac{p - r}{q}$

Steps to Find the LCD of Two or More Rational Expressions

1. Factor all denominators completely.
2. The LCD is the product of unique prime factors from the denominators, where each factor is raised to the highest power to which it appears in any denominator.

Steps to Add or Subtract Rational Expressions

1. Factor the denominator of each rational expression.
2. Identify the LCD.
3. Rewrite each rational expression as an equivalent expression with the LCD as its denominator.
4. Add or subtract the numerators, and write the result over the common denominator.
5. Simplify if possible.

Steps to Simplify a Complex Fraction—Method I

1. Add or subtract expressions in the numerator to form a single fraction. Add or subtract expressions in the denominator to form a single fraction.
2. Divide the rational expressions from step 1 by multiplying the numerator of the complex fraction by the reciprocal of the denominator of the complex fraction.
3. Simplify to lowest terms, if possible.

Steps to Simplify a Complex Fraction—Method II

1. Multiply the numerator and denominator of the complex fraction by the LCD of *all* individual fractions within the expression.
2. Apply the distributive property, and simplify the numerator and denominator.
3. Simplify to lowest terms, if possible.

Steps to Simplify a Complex Fraction—Method I

1. Add or subtract expressions in the numerator to form a single fraction. Add or subtract expressions in the denominator to form a single fraction.
2. Divide the rational expressions from step 1 by multiplying the numerator of the complex fraction by the reciprocal of the denominator of the complex fraction.
3. Simplify to lowest terms, if possible.

Steps to Simplify a Complex Fraction—Method II

1. Multiply the numerator and denominator of the complex fraction by the LCD of *all* individual fractions within the expression.
2. Apply the distributive property, and simplify the numerator and denominator.
3. Simplify to lowest terms, if possible.

Definition of a Rational Equation

An equation with one or more rational expressions is called a **rational equation**.

The following equations are rational equations:

$$\frac{1}{2}x + \frac{1}{3} = \frac{1}{4}x \qquad \frac{3}{5} + \frac{1}{x} = \frac{2}{3} \qquad 3 - \frac{6w}{w+1} = \frac{6}{w+1}$$

Steps for Solving a Rational Equation

1. Factor the denominators of all rational expressions. Identify any values of the variable for which any expression is undefined.
2. Identify the LCD of all terms in the equation.
3. Multiply both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check the potential solutions in the original equation. Note that any value from step 1 for which the equation is undefined cannot be a solution to the equation.

Avoiding Mistakes:

Variables in algebra are case-sensitive. Therefore, *V* and *v* are different variables, and *M* and *m* are different variables.

Definition of Ratio and Proportion

1. The **ratio** of a to b is $\frac{a}{b}$ $(b \neq 0)$ and can also be expressed as $a:b$ or $a \div b$.
2. An equation that equates two ratios or rates is called a **proportion**. Therefore, if $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ is a proportion.

TIP: For any proportion

$$\frac{a}{b} = \frac{c}{d} \qquad b \neq 0, d \neq 0$$

the cross products of terms are equal. Hence, $ad = bc$. Finding the cross product is a quick way to clear fractions in a proportion.* Consider Example 1:

$$\frac{5}{19} = \frac{95}{y}$$

$5y = (19)(95)$ Equate the cross products.

$5y = 1805$

$y = 361$

*It is important to realize that this method is only valid for proportions.