Continuity Practice … Set 4

Example Which of the following functions are continuous on the interval $(0, \infty)$:

$$f(x) = \frac{x^3 + x - 1}{x + 2}, \qquad g(x) = \frac{x^2 + 3}{\cos x}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}, \qquad k(x) = |\sin x|.$$

Continuity Practice … Set 4

Answers

Example Which of the following functions are continuous on the interval $(0, \infty)$:

$$f(x) = \frac{x^3 + x - 1}{x + 2}, \qquad g(x) = \frac{x^2 + 3}{\cos x}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}, \qquad k(x) = |\sin x|.$$

Since $f(x)$ is a rational function, it is continuous everywhere except at $x = -2$, Therefore it is continuous on the interval $(0, \infty)$.
By Theorem 2 and the continuity of polynomials and trigonometric functions, $g(x)$ is continuous except where $\cos x = 0$. Since $\cos x = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, we have $g(x)$ is not continuous on $(0, \infty)$.
By theorems 2 and 3, $h(x)$ is continuous everywhere except at $x = 2$. In fact $x = 2$ is not in the domain of this function. Hence the function is not continuous on the interval $(0, \infty)$.
Since $k(x) = |\sin x| = F(G(x))$, where $G(x) = \sin x$ and $F(x) = |x|$, we have that $k(x)$ is continuous everywhere on its domain since both F and G are both continuous everywhere on their domains. Its not difficult to see that the domain of k is all real numbers, hence k is continuous everywhere. (What does its graph look like?)

Continuity Practice … Set 4

Example Which of the following functions have a removable discontinuity at $x = 2$?:

$$f(x) = \frac{x^3 + x - 1}{x - 2}, \qquad g(x) = \frac{x^2 - 4}{x - 2}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}.$$

Continuity Practice … Set 4

Answers

Example Which of the following functions have a removable discontinuity at $x = 2$?:

$$f(x) = \frac{x^3 + x - 1}{x - 2}, \qquad g(x) = \frac{x^2 - 4}{x - 2}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}.$$

$\lim_{x\to 2} f(x)$ does not exist, since $\lim x \to 2(x^3 + x - 1) = 9$ and $\lim x \to 2(x - 2) = 0$. Therefore the discontinuity is not removable.

$\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x + 2) = 4$. Therefore the discontinuity at $x = 2$ is removable by defining a piecewise function:

$$g_1(x) = \begin{cases} g(x) & x \neq 2 \\ 4 & x = 2 \end{cases}$$

$\lim_{x\to 2} h(x)$ does not exist, since $\lim_{x\to 2}(\sqrt{x^2 + 1}) = \sqrt{(5)}$ and $\lim x \to 2(x - 2) = 0$. Therefore the discontinuity is not removable.

Continuity Practice … Set 4

Example Find the domain of the following function and use Theorems 1, 2 and 3 to show that it is continuous on its domain:

$$k(x) = \frac{\sqrt[3]{\cos x}}{x - 10}.$$

Continuity Practice … Set 4

Answers

Example Find the domain of the following function and use Theorems 1, 2 and 3 to show that it is continuous on its domain:

$$k(x) = \frac{\sqrt[3]{\cos x}}{x - 10}.$$

The domain of this function is all values of x except $x = 10$, since $\cos x$ is defined everywhere as is the cubed root function. Theorem 1 says that the cosine function is continuous everywhere and theorem 3 says that $f(x) = \sqrt[3]{\cos x}$ is continuous for all real numbers since the cubed root function is continuous everywhere. Now we see from Theorem 2 that $k(x) = \frac{f(x)}{g(x)}$ is continuous everywhere except where $g(x) = x - 10 = 0$, that is at $x = 10$.

Continuity Practice … Set 4

Example Evaluate the following limits:

$$\lim_{x \to \pi} \sqrt[3]{2 + \cos x} \qquad\qquad \lim_{x \to \frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x - \frac{\pi}{2}}$$

Continuity Practice … Set 4

Answers

Example Evaluate the following limits:

$$\lim_{x\to\pi} \sqrt[3]{2+\cos x} \qquad\qquad \lim_{x\to\frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x-\frac{\pi}{2}}$$

Since $G(x) = 2+\cos x$ and $F(x) = \sqrt[3]{x}$ are continuous everywhere, we have $F(Gx))$ is continuous on its domain and we can calculate the first limit by evaluation:

$$\lim_{x\to\pi} \sqrt[3]{2+\cos x} = \sqrt[3]{2+\cos\pi} = \sqrt[3]{2-1} = 1.$$

As above, we have $\sqrt[3]{\sin x}$ is continuous on its domain, therefore $\lim_{x\to\frac{\pi}{2}} \sqrt[3]{\sin x} = \sqrt[3]{\sin\frac{\pi}{2}} = 1$. Since $\lim_{x\to\frac{\pi}{2}}(x-\frac{\pi}{2}) = 0$, we have $\frac{\sqrt[3]{\sin x}}{x-\frac{\pi}{2}}$ approaches ∞ in absolute value as x approaches $\frac{\pi}{2}$. As $x \to \frac{\pi}{2}^-$, $\sin(x) > 0$, hence $\sqrt[3]{\sin x} > 0$. As $x \to \frac{\pi}{2}^-$, $x - \frac{\pi}{2} < 0$, therefore the quotient has negative values and

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x-\frac{\pi}{2}} = -\infty.$$

Continuity Practice … Set 4

Example What is the domain of the following function and what are the (largest) intervals on which it is continuous?

$$g(x) = \frac{1}{\sqrt{1-\sqrt{x}}}.$$

Continuity Practice … Set 4

Answers

Example What is the domain of the following function and what are the (largest) intervals on which it is continuous?

$$g(x) = \frac{1}{\sqrt{1-\sqrt{x}}}.$$

The domain of this function is all x where $\sqrt{1-\sqrt{x}} \neq 0$, i.e. all x where $x \neq 1$. By theorems 3 and 2, the function is continuous everywhere on its domain, therefore it is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$.

Continuity Practice … Set 4

Example use the intermediate value theorem to show that there is a root of the equation in the specified interval:

$$\sqrt[3]{x} = 1 - x \qquad (0, 1).$$

Continuity Practice … Set 4

Answers

Example use the intermediate value theorem to show that there is a root of the equation in the specified interval:

$$\sqrt[3]{x} = 1 - x \qquad (0, 1).$$

Let $g(x) = \sqrt[3]{x} - 1 + x$. We have $g(0) = -1 < 0$ and $g(1) = 1 > 0$. therefore by the intermediate value theorem, there is some number c with $0 < c < 1$ for which $g(c) = 0$. That is

$$\sqrt[3]{c} = 1 - c$$

as desired.