Differentiability… Facts 2

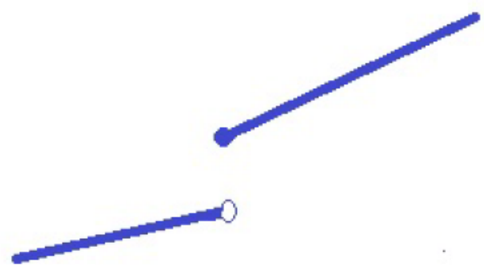
LESSON 2.6: *Differentiability:*

A function is differentiable at a point if it has a derivative there. In other words:
The function f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{exists.}$$

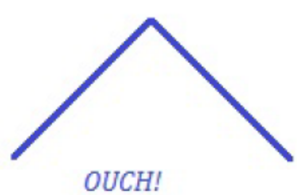
Thus, the graph of f has a non-vertical tangent line at $(x, f(x))$. The value of the limit and the slope of the tangent line are the derivative of f at x_0.
A function can fail to be differentiable at point if:

1. The function is not continuous at the point.

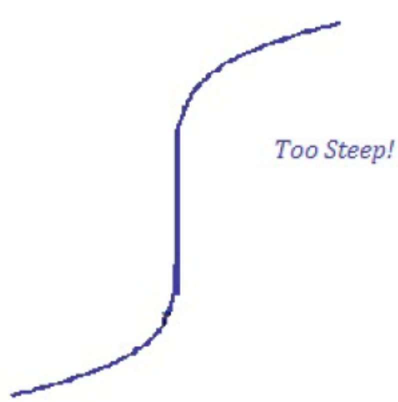


How can you make a tangent line here?

2. The graph has a sharp corner at the point.

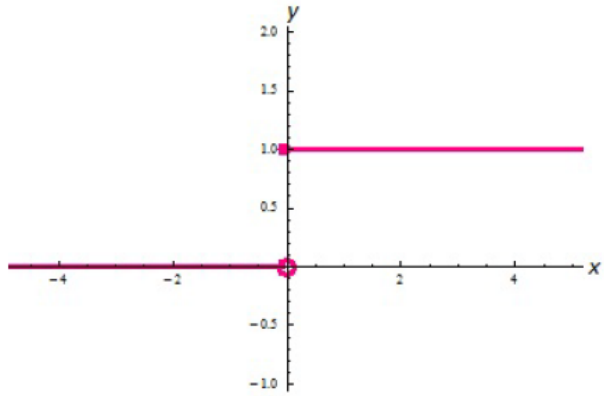


3. The graph has a vertical line at the point.



Differentiability… Facts 2

Example 1:



$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

H is not continuous at 0, so it is not differentiable at 0.

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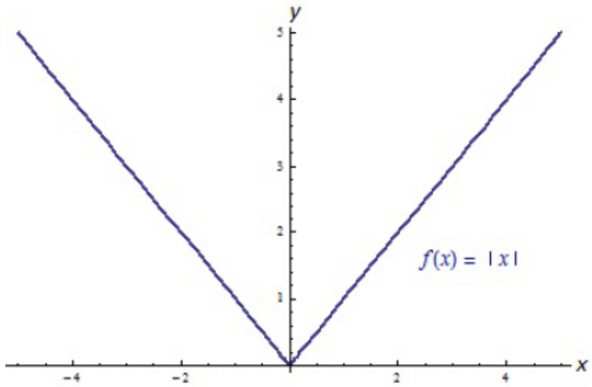
Theorem 2.1: A differentiable function is continuous:

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Example 2:



$$f(x) = |\, x \,| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

At $x = 0$, there is a corner at $(0, 0)$. Picture is different to the left and right of $(0, 0$.
Suggests we try left- and right-hand limits.

$$\lim_{h \to 0_+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0_+} \frac{h - 0}{h} \qquad \text{cancellation of } h \text{ okay, since } h \neq 0 \text{ for limit at } 0$$
$$= \lim_{h \to 0_+} 1$$
$$= 1$$

But

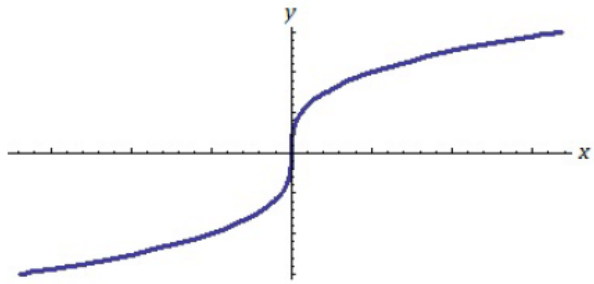
$$\lim_{h \to 0_-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0_-} \frac{-h - 0}{h} \qquad \text{cancellation of } h \text{ okay, since } h \neq 0 \text{ for limit at } 0$$
$$= \lim_{h \to 0_-} -1$$
$$= -1$$

Since the left- and right-hand limits do not agree,

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

Does not exits, and so $|\, x \,|$ is not differentiable at $x = 0$.

Example 3:



$$g(x) = x^{1/3}$$

The graph is smooth at $x = 0$, but does appear to have a vertical tangent.

$$\lim_{h \to 0} \frac{(0+h)^{1/3} - 0^{1/3}}{h} = \lim_{h \to 0} \frac{(h)^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}$$

As $h \to 0$, the denominator becomes small, so the fraction grows without bound. Hence g is not differentiable at $x = 0$.

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Example 4:

$$g(x) = \begin{cases} x+1 & x \leq 1 \\ 3x-1 & x > 1 \end{cases}$$