Trig Rules … Set 1

Derivatives of Trig Functions

There are a lot of trigonometry problems in calculus and on the AP Calculus Exam. If you're not sure of your trig, you should definitely go to the Appendix and review the unit on Prerequisite Math. You'll need to remember your trig formulas, the values of the special angles, and the trig ratios, among other stuff.

In addition, angles are *always* referred to in radians. You can forget all about using degrees.

You should know the derivatives of all six trig functions. The good news is that the derivatives are pretty easy, and all you have to do is memorize them. Because the AP exam might ask you about this, though, let's use the definition of the derivative to figure out the derivative of $\sin x$.

$$\text{If } f(x) = \sin x, \text{ then } f(x+h) = \sin(x+h).$$

Substitute this into the definition of the derivative:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Remember that $\sin(x + h) = \sin x \cos h + \cos x \sin h$. Now simplify it:

$$\lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

Next, rewrite this as:

$$\lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} = \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

Next, use some of the trigonometric limits that you memorized back in Chapter 3. Specifically:

$$\lim_{h \to 0} \frac{(\cos h - 1)}{h} = 0 \text{ and } \lim_{h \to 0} \frac{\sin h}{h} = 1$$

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This gives you:

$$\lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}=\sin x(0)+\cos x(1)=\cos x$$

$$\frac{d}{dx}\sin x=\cos x$$

Example 1: Find the derivative of $\sin\left(\frac{\pi}{2}-x\right)$.

$$\frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right)=\cos\left(\frac{\pi}{2}-x\right)(-1)=-\cos\left(\frac{\pi}{2}-x\right)$$

Use some of the rules of trigonometry you remember from last year. Because

$$\sin\left(\frac{\pi}{2}-x\right)=\cos x \text{ and } \cos\left(\frac{\pi}{2}-x\right)=\sin x,$$

you can substitute into the above expression and get:

$$\frac{d}{dx}\cos x=-\sin x$$

Now, let's derive the derivatives of the other four trigonometric functions.

Example 2: Find the derivative of $\frac{\sin x}{\cos x}$.

Use the Quotient Rule:

$$\frac{d}{dx}\frac{\sin x}{\cos x}=\frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{(\cos x)^2}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

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Because $\frac{\sin x}{\cos x} = \tan x$, you should get:

$$\frac{d}{dx}\tan x = \sec^2 x$$

Example 3: Find the derivative of $\frac{\cos x}{\sin x}$.

Use the Quotient Rule:

$$\frac{d}{dx}\frac{\cos x}{\sin x} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-(\cos^2 x + \sin^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

Because $\frac{\cos x}{\sin x} = \cot x$, you get: $\frac{d}{dx}\cot x = -\csc^2 x$.

Example 4: Find the derivative of $\frac{1}{\cos x}$.

Use the Reciprocal Rule.

$$\frac{d}{dx}\frac{1}{\cos x} = \frac{-1}{(\cos x)^2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x$$

Because $\frac{1}{\cos x} = \sec x$, you get:

$$\frac{d}{dx}\sec x = \sec x \tan x$$

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Example 5: Find the derivative of $\frac{1}{\sin x}$.

You get the idea by now.

$$\frac{d}{dx}\frac{1}{\sin x} = \frac{-1}{(\sin x)^2}(\cos x) = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x}\frac{\cos x}{\sin x} = -\csc x \cot x$$

Because $\frac{1}{\sin x} = \csc x$, you get:

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

There you go. We have now found the derivatives of all six of the trigonometric functions. (A chart of them appears at the end of the book.) Now memorize them. You'll thank us later.

Let's do some more examples:

Example 6: Find the derivative of $\sin(5x)$.

$$\frac{d}{dx}\sin(5x) = \cos(5x)(5) = 5\cos(5x)$$

Example 7: Find the derivative of $\sec(x^2)$.

$$\frac{d}{dx}\sec(x^2) = \sec(x^2)\tan(x^2)(2x)$$

Example 8: Find the derivative of $\csc(x^3 - 5x)$.

$$\frac{d}{dx}\csc(x^3 - 5x) = -\csc(x^3 - 5x)\cot(x^3 - 5x)(3x^2 - 5).$$

These derivatives are almost like formulas. You just follow the pattern and use the Chain Rule when appropriate.

Here are some solved problems. Do each problem, covering the answer first, then checking your answer.

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Problem 1. Find $f'(x)$ if $f(x) = \sin(2x^3)$.

Answer: Follow the rule: $f'(x) = \cos(2x^3)6x^2$

Problem 2. Find $f'(x)$ if $f(x) = \cos\left(\sqrt{3x}\right)$.

Answer: $f'(x) = -\sin\left(\sqrt{3x}\right)\left[\frac{1}{2}(3x)^{-\frac{1}{2}}(3)\right] = \frac{-3\sin\left(\sqrt{3x}\right)}{2\sqrt{3x}}$

Problem 3. Find $f'(x)$ if $f(x) = \tan\left(\frac{x}{x+1}\right)$.

Answer: $f'(x) = \sec^2\left(\frac{x}{x+1}\right)\left[\frac{(x+1)-x}{(x+1)^2}\right] = \left(\frac{1}{(x+1)^2}\right)\sec^2\left(\frac{x}{x+1}\right)$

Problem 4. Find $f'(x)$ if $f(x) = \csc(x^3 + x + 1)$.

Answer: Follow the rule: $f'(x) = -\csc(x^3 + x + 1)\cot(x^3 + x + 1)(3x^2 + 1)$

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1. Find $\frac{dy}{dx}$ if $y = \sin^2 x$.

2. Find $\frac{dy}{dx}$ if $y = \cos x^2$.

3. Find $\frac{dy}{dx}$ if $y = (\tan x)(\sec x)$.

4. Find $\frac{dy}{dx}$ if $y = \cot 4x$.

5. Find $\frac{dy}{dx}$ if $y = \sqrt{\sin 3x}$.

6. Find $\frac{dy}{dx}$ if $y = \frac{1+\sin x}{1-\sin x}$.

7. Find $\frac{dy}{dx}$ if $y = \csc^2 x^2$.

8. Find $\frac{dy}{dx}$ if $y = 2 \sin 3x \cos 4x$.

9. Find $\frac{d^4 y}{dx^4}$ if $y = \sin 2x$.

10. Find $\frac{dy}{dx}$ if $y = \sin t - \cos t$ and $t = 1 + \cos^2 x$.

11. Find $\frac{dy}{dx}$ if $y = \left(\frac{\tan x}{1-\tan x}\right)^2$.

12. Find $\frac{dr}{d\theta}$ if $r = \sec\theta \tan 2\theta$.

13. Find $\frac{dr}{d\theta}$ if $r = \cos(1 + \sin\theta)$.

14. Find $\frac{dr}{d\theta}$ if $r = \frac{\sec\theta}{1+\tan\theta}$.

15. Find $\frac{dy}{dx}$ if $y = \left(1 + \cot\left(\frac{2}{x}\right)\right)^{-2}$.

16. Find $\frac{dy}{dx}$ if $y = \sin\left(\cos\left(\sqrt{x}\right)\right)$.