Implicit Differentiation … Set 2

Implicit Differentiation

The goal of this section is to learn how to compute the derivative when there is not an *explicit* equation available for the original function. Sometimes a function is defined *implicitly* as part of an equation. For example, suppose $y = f(x)$ is defined by the equation

$$xy^3 + e^{xy} = 1 .$$

There is no way to isolate y in this equation. Nevertheless, there is a curve of points in the xy-plane that satisfy this equation (note that $(0,0)$ is one such point). Implicit differentiation can be used to find the derivative (slope of the tangent line to the curve) at a given point.

Example 1: Suppose that $x^2 + y^2 = 1$ (the unit circle). Use implicit differentiation to find a formula for $\frac{dy}{dx}$.

Answer: First replace y by $y(x)$, treating y as a function of x. This gives $x^2 + (y(x))^2 = 1$. Now differentiate both sides of the equation with respect to x. Using the chain rule on the second term, with $y(x)$ as the "inside," we find

$$2x + 2(y(x))^1 \cdot \frac{dy}{dx} = 0 .$$

Now solve for $\frac{dy}{dx}$ (think of it as a variable that you want to isolate). It is easier to replace $y(x)$ by y when solving. We have

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies 2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y} .$$

Note that this is the same answer we obtained in class by first solving for y and then differentiating explicitly. The formula $\frac{dy}{dx} = -\frac{x}{y}$ gives us the slope of the tangent line to the unit circle at the point (x, y). For example, at the points $(0, \pm 1)$, we find $m = 0$ (horizontal tangent lines), while at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ we have $m = -1$. These make sense if we draw the unit circle.

Exercise 1: If $xy + y^4 + 3 = 0$, use implicit differentiation to find $\frac{dy}{dx}$.

Exercise 2: Suppose that $\sin(y) + e = e^x + xy^2$. Find $\frac{dy}{dx}$ and use it to find the equation of the tangent line at the point $(1, 0)$.

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Derivatives of Inverse Trig Functions

Implicit differentiation can be used to determine the derivatives of some key inverse functions. For example, consider the inverse sine function $y = \sin^{-1} x$. Recall that this function outputs the angle y whose sine value is x. Thus, we have $\sin^{-1}(1) = \pi/2$ because $\sin(\pi/2) = 1$.

Here is how we use implicit differentiation to compute $\frac{dy}{dx}$. First note that since $y = \sin^{-1} x$, we have $\sin(y) = x$ by definition of the inverse sine function. Then

$$\begin{aligned} \sin(y) \;=\; x \qquad &\Longrightarrow \qquad \cos(y) \cdot \frac{dy}{dx} \;=\; 1 \\ &\Longrightarrow \qquad \frac{dy}{dx} \;=\; \frac{1}{\cos(y)} \;=\; \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

Here the final step follows from the fundamental trig identity and the fact that $\sin y = x$, as

$$\cos^2 y + \sin^2 y \;=\; 1 \quad \Longrightarrow \quad \cos^2 y + x^2 \;=\; 1 \quad \Longrightarrow \quad \cos y = \sqrt{1-x^2}\,.$$

We choose the positive square root because $y = \sin^{-1} x$ is restricted to the interval $-\pi/2 \le y \le \pi/2$. Since $\cos y$ is positive in the first and fourth quadrants, we are justified in choosing the positive square root. The formula for the derivative of $y = \cos^{-1} x$ is found in a similar fashion. Here are two new formulas to add to the list:

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) \;=\; \frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) \;=\; -\frac{1}{\sqrt{1-x^2}}\,.}$$

Exercise 3: Suppose that $f(x) = \sin^{-1}(\sqrt{x}\,)$. Use the formula above to compute and simplify $f'(x)$.

Exercise 4: Find the formula for $\frac{d}{dx}\left(\tan^{-1} x\right)$ using the trig identity $\tan^2 y + 1 = \sec^2 y$.

Exercise 5: Use the formula from the previous exercise to find $\dfrac{d}{dx}\left(\tan^{-1}(e^x)\right)$.