Critical Values … Set 3

Problems

Find all critical points of the given functions.

1. $f(x) = x^3 - 6x + 1$
2. $f(x) = x^3 + 6x + 1$
3. $f(x) = 3x^5 - 5x^3$
4. $f(x) = e^x - 10x$
5. $y = x\ln(x)$, $(x > 0)$
6. $y = xe^{-3x}$
7. $f(x) = x + \frac{1}{x}$
8. $f(x) = 3x^4 - 4x^3 + 6$
9. $y = (x^2 - 4)^7$
10. $y = (x^3 - 8)^7$
11. $f(x) = x^2\ln(x)$
12. $y = (x + 1)^5$
13. $y = \frac{x}{x^2+1}$
14. $y = \sqrt{x^2 + 1}$
15. $g(x) = (4x^2 + 1)^7$

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Example 1:

Find all local extrema of $f(x) = x^3 - 3x^2 - 9x + 5$.

In Example 1 on the previous page we found $f'(x) = 3x^2 - 6x - 9$ and that the critical points occur at $(-1, 10)$ and $(3, -22)$. To apply the second derivative test we must first compute the second derivative;

$$f''(x) = 6x - 6$$

Then we simply take the points $x = -1$ and $x = 3$ and plug them into $f''(x)$.

$x = -1$:

$$f''(-1) = 6(-1) - 6 = -6 - 6 = -12 < 0$$
$\implies$ Local Maximum

$x = 3$:

$$f''(3) = 6(3) - 6 = 18 - 6 = 12 > 0$$
$\implies$ Local Minimum

So the function $f(x)$ has a local maximum at the point $(-1, 10)$ and a local minimum at the point $(3, -22)$.

Example 2:

Find all local extrema of $f(t) = e^{-3t} + 2$.

In Example 2 on the previous page we found $f'(t) = -3e^{-3t} + 2$ and that the only critical point occurs at $t = \frac{1}{3}\ln(1.5)$. To apply the second derivative test we must first compute the second derivative;

$$f''(t) = 9e^{-3t}$$

Now we simply plug in $t = \frac{1}{3}\ln(1.5)$ into $f''(t)$.

$$f''\left(\tfrac{1}{3}\ln(1.5)\right) = 9e^{-3(\ln(1.5)/3)} = 9e^{-\ln(1.5)} = 9e^{\ln(2/3)} = 9(2/3) = 6 > 0 \implies \text{Local Minimum.}$$

So the function $f(t)$ has a local minimum at the point $t = \frac{1}{3}\ln(1.5)$.

Example 3:

Find all local extrema of $y = x + \dfrac{1}{x}$.

First we calculate the critical point(s) of y. Differentiating gives $y' = 1 - x^{-2}$. Solving $y' = 0$ gives

$$y' = 1 - x^{-2} = 0 \implies 1 = x^{-2} \implies 1 = \frac{1}{x^2} \implies 1 = x^2 \implies x = -1 \text{ or } x = 1$$

Next we find the second derivative,

$$y'' = 2x^{-3} = \frac{2}{x^3}$$

Finally we plug both $x = -1$ and $x = 1$ into y''.

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$x=-1$:

$$y''(-1)=\frac{2}{(-1)^3}=\frac{2}{-1}=-2<0\Longrightarrow \text{Local Maximum}$$

$x=1$:

$$y''(1)=\frac{2}{1^3}=\frac{2}{1}=2>0\Longrightarrow \text{Local Minimum.}$$

Thus y has a local maximum at the point $(-1,y(-1))=(-1,-2)$ and a local minimum at the point $(1,y(1))=(1,2)$.

Problems

Go back and classify the critical points in questions $1-15$ on page 1 as maximum, minimum or neither (points of inflection).

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Answers

Critical Points

1. $f'(x) = 3x^2 - 6$
   C. Pts:
   $(\sqrt{2}, 1 - 4\sqrt{2}) \approx (1.414, -4.657)$
   $(-\sqrt{2}, 1 + 4\sqrt{2}) \approx (-1.414, 6.657)$

2. $f'(x) = 3x^2 + 6$
   C. Pts:
   None.

3. $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$
   C. Pts:
   $(0, 0)$
   $(1, -2)$
   $(-1, 2)$

4. $f'(x) = e^x - 10$
   C. Pts:
   $(\ln(10), 10(1 - \ln(10))) \approx (2.302, -13.026)$

5. $y' = \ln(x) + 1$
   C. Pts:
   $\left(\frac{1}{e}, -\frac{1}{e}\right) \approx (0.368, -0.368)$

6. $y' = e^{-3x} - 3xe^{-3x}$
   C. Pts:
   $\left(\frac{1}{3}, \frac{1}{3e}\right)$

7. $f'(x) = 1 - x^{-2}$
   C. Pts:
   $(-1, -2)$
   $(1, 2)$

8. $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$
   C. Pts:
   $(0, 6)$
   $(1, 5)$

9. $y' = 14x\left(x^2 - 4\right)^6$
   C. Pts:
   $(0, -16384)$
   $(2, 0)$
   $(-2, 0)$

10. $y' = 21x^2(x^3 - 8)^6$
    C. Pt.s
    $(0, -2097152)$
    $(2, 0)$

11. $f'(x) = 2x\ln(x) + x$
    C. Pts:
    $\left(\frac{1}{\sqrt{e}}, 1\frac{1}{2e}\right)$

12. $y' = 5(x + 1)^4$
    C.Pts:
    $(-1, 0)$

13. $y' = (x^2 + 1)^{-1} - 2x^2(x^2 + 1)^{-2}$
    C. Pts:
    $\left(-1, -\frac{1}{2}\right)$
    $\left(1, \frac{1}{2}\right)$

14. $y' = x\sqrt{x^2 + 1}^{-1/2}$
    C. Pts:
    $(0, 1)$

15. $g'(x) = 56x(4x^2 + 1)^6$
    C. Pts:
    $(0, 1)$

Local Extrema

1. $f''(x) = 6x$
   $(\sqrt{2}, 1 - 4\sqrt{2})$ Min.
   $(-\sqrt{2}, 1 + 4\sqrt{2})$ Max.

2. $f''(x) = 6x$
   None.

3. $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$
   $(0, 0)$ Neither
   $(1, -2)$ Min.
   $(-1, 2)$ Max.

4. $f''(x) = e^x$
   $(\ln(10), 10(1 - \ln(10)))$ Min.

5. $y'' = x^{-1}$
   $\left(\frac{1}{e}, -\frac{1}{e}\right)$ Min.

6. $y'' = -6e^{-3x} + 9xe^{-3x}$
   $\left(\frac{1}{3}, \frac{1}{3e}\right)$ Max.

7. $f''(x) = 2x^{-3}$
   $(-1, -2)$ Min.
   $(1, 2)$ Max.

8. $f''(x) = 36x^2 - 24x = 12x(3x - 2)$
   $(0, 6)$ Neither
   $(1, 5)$ Min.

9. $y'' = 14\left(x^2 - 4\right)^6 + 168x^2(x^2 - 4)^5$
   $(0, -16384)$ Min.
   $(2, 0)$ Neither
   $(-2, 0)$ Neither

10. $y'' = 42x(x^3 - 8)^6 + 378x^4(x^3 - 8)^5$
    $(0, -2097152)$ Neither
    $(2, 0)$ Neither

11. $f''(x) = 2\ln(x) + 3$
    $\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$ Min.

12. $y'' = 20(x + 1)^3$
    $(-1, 0)$ Neither

13. $y'' = -2x(x^2 + 1)^{-2} - 4x(x^2 + 1)^{-2} + 8x^3(x^2$
    $1)^{-3}$
    $\left(-1, -\frac{1}{2}\right)$ Min.
    $\left(1, \frac{1}{2}\right)$ Max.

14. $y'' = \sqrt{x^2 + 1}^{-1/2} - x^2\sqrt{x^2 + 1}^{-1/2}$
    $(0, 1)$ Min.

15. $g''(x) = 56(4x^2 + 1)^6 + 3088x^2(4x^2 + 1)^5$
    $(0, 1)$ Min.