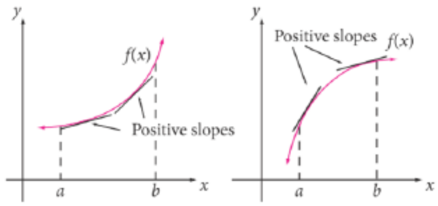
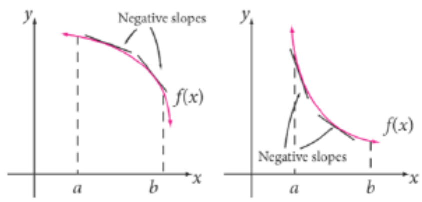
2nd Derivative Test … Set 2

The First Derivative and Behavior of Functions

Definition [Increasing/Decreasing]: A function $f(x)$ is **increasing** on an interval (a, b) if its graph rises (from left to right) through (a, b).



Similarly, $f(x)$ is **decreasing** on an interval (a, b) if its graph falls through (a, b).



The First Derivative $f'(x)$ and the Increasing/Decreasing Behavior of $f(x)$

Suppose $f(x)$ is continuous and differentiable on interval (a, b). Then

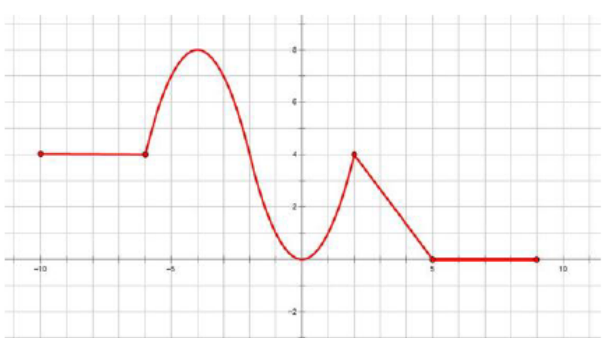
1. $f(x)$ is increasing on (a, b) if and only if $f'(x) > 0$ for each x in (a, b).
2. $f(x)$ is decreasing on (a, b) if and only if $f'(x) < 0$ for each x in (a, b).
3. $f(x)$ is constant on (a, b) if and only if $f'(x) = 0$ for each x in (a, b).

Example: Determine the intervals on which the function below is

Increasing:

Decreasing:

Constant:



2^{nd} Derivative Test … Set 2

Definition [Critical Values]: Suppose $f(x)$ is defined at $x = c$. If either $f'(c) = 0$ or $f'(c)$ is undefined, then c is called a **critical value** of $f(x)$.

*These are the points where the behavior of f *potentially* changes!

2nd Derivative Test … Set 2

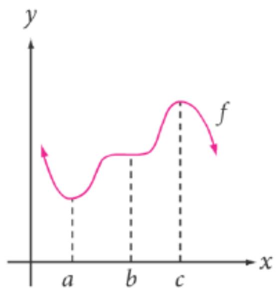
The Procedure for Constructing a Sign Chart

1. Find all critical values of $f(x)$.
2. Draw a number line with points at the critical values.
3. In each interval (between the critical values), choose a 'test number' a.
   (a) If $f'(a) > 0$ then $f(x)$ is increasing on the interval.
   (b) If $f'(a) < 0$ then $f(x)$ is decreasing on the interval.

Example: Find the intervals on which $f(x) = x^3 - 9x^2 + 24x$ is increasing/decreasing.

2^{nd} Derivative Test … Set 2

Definition [Relative Extrema]: A point $(c, f(c))$ is called a **relative maximum** if there is an interval I in the domain of $f(x)$ such that $f(c) \geq f(x)$ for all x in I.



Similarly, $(a, f(a))$ is called a **relative minimum** if there is an interval I in the domain of $f(x)$ such that $f(a) \leq f(x)$ for all x in I.

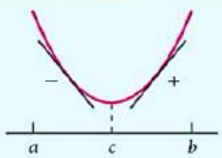
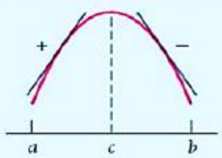
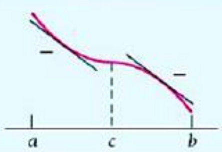
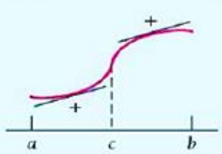
The term **relative extrema** is used to describe points that are either relative maxima or relative minima.

2^{nd} Derivative Test … Set 2

The First Derivative Test

If c is a critical value of $f(x)$, and

1. $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c, then $(c, f(c))$ is a relative maximum.
2. $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c, then $(c, f(c))$ is a relative minimum.

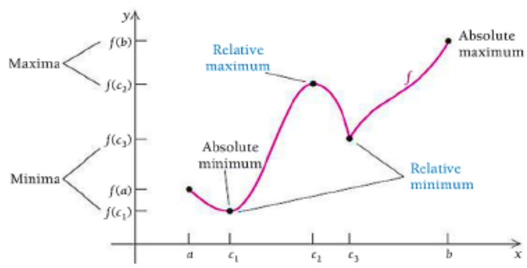
Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	−	+	Decreasing on (a, c); increasing on (c, b)
	Relative maximum	+	−	Increasing on (a, c); decreasing on (c, b)
	No relative maxima or minima	−	−	Decreasing on (a, b)
	No relative maxima or minima	+	+	Increasing on (a, b)

2^{nd} Derivative Test … Set 2

Example: Find the relative extrema for $f(x) = x^3 - 9x^2$.

2nd Derivative Test … Set 2

Definition [Absolute Extrema]: A point $(c, f(c))$ is called an **absolute maximum** if $f(c) \geq f(x)$ for all x in the domain of $f(x)$. A point $(a, f(a))$ is called an **absolute minimum** if $f(a) \geq f(x)$ for all x in the domain of $f(x)$. The term **absolute extrema** is used to describe points that are either absolute maxima or absolute minima.



Note: If the interval on which we are finding absolute extrema is closed, like [a,b], then f(a) and f(b) are also candidates for being absolute extrema.

2^{nd} Derivative Test … Set 2

Example: Find, if they exist, all relative and absolute extrema of the function $f(x) = (x-4)^{2/7} + 3$ on the following intervals:

i) $[0, 6]$

ii) $(0, 6]$

iii) $[0, 6)$

iv) $(0, 6)$

2^{nd} Derivative Test … Set 2

Example: A box with a square base and an open top is to be made from a square piece of cardboard by cutting out four squares of equal size from the corners and folding up the sides. The function relating the volume V (in cubic inches) and the size x (in inches) of the corner cuts is

$$V(x) = 4x^3 - 60x^2 + 225x$$

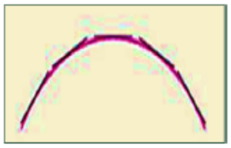
What size should the corner cuts be so that the volume of the box is as large as possible?

2nd Derivative Test … Set 2

The Second Derivative and Behavior of Functions

Definition [Concavity]: If the graph of $f(x)$ opens upward at $x = c$, then $f(x)$ is said to be **concave upward** at $x = c$.

Similarly, if the graph of $f(x)$ opens downward at $x = c$, then $f(x)$ is said to be **concave downward** at $x = c$.



The Second Derivative $f''(x)$ and the Concavity of $f(x)$

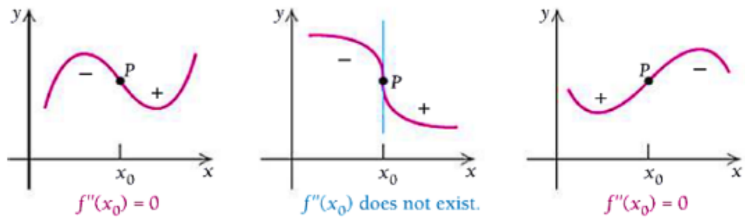
Suppose $f(x)$ is continuous and twice-differentiable on interval (a, b). Then

1. $f(x)$ is concave upward on (a, b) if and only if $f''(x) > 0$ for each x in (a, b).
2. $f(x)$ is concave downward on (a, b) if and only if $f''(x) < 0$ for each x in (a, b).

Definition [Hypercritical Values]: Suppose $f(x)$ is defined at $x = c$. If either $f''(c) = 0$ or $f''(c)$ is undefined, then c is called a **hypercritical value** of $f(x)$.

*These are the points where the concavity of f *potentially* changes!

Definition [Inflection Points]: Hypercritical values where $f(x)$ changes concavity are called **points of inflection**. We find them by finding the points where the sign of f'' changes.



2^{nd} Derivative Test … Set 2

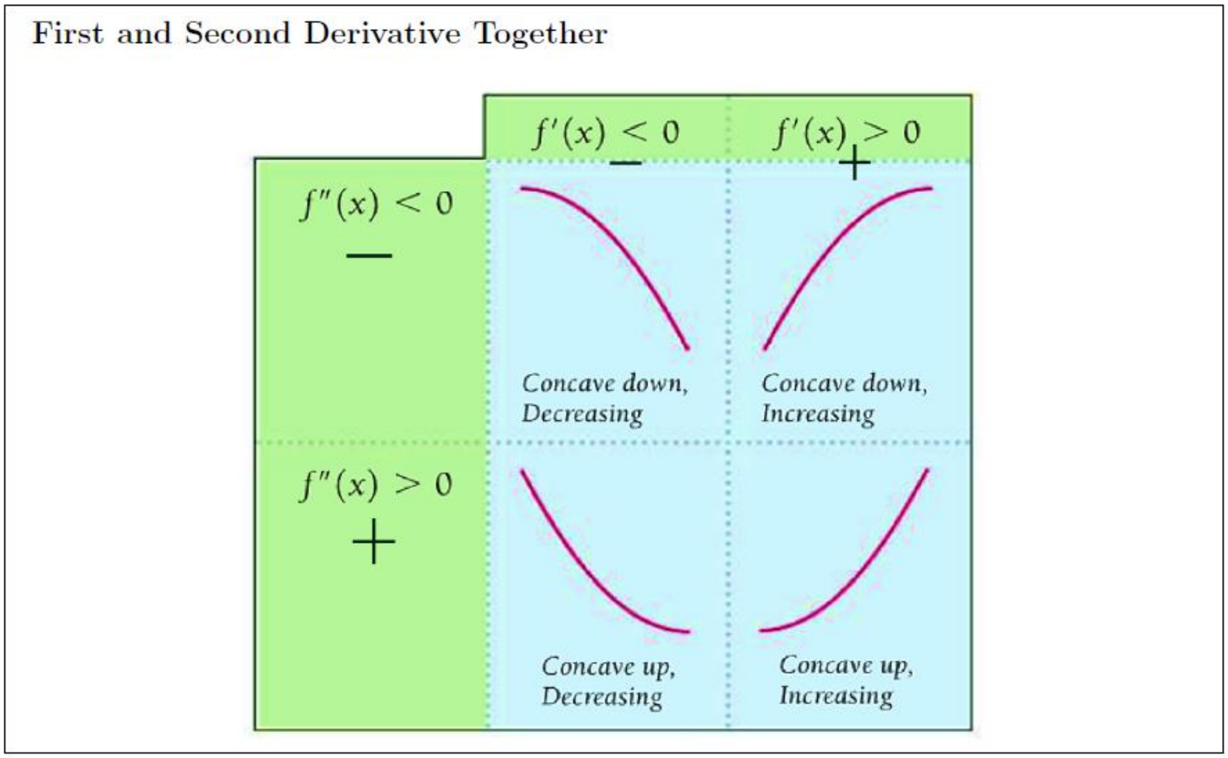
Example: Find all hypercritical values of $f(x)$, state whether they are points of inflection, and find the intervals where $f(x)$ is concave up/down.

a. $f(x) = x^4 + 4x^3$

b. $f(x) = \frac{1}{5}x^6 - \frac{3}{2}x^4 - 12x^2$

c. $f(x) = (x-1)^{2/3}$

2^{nd} Derivative Test … Set 2



2nd Derivative Test … Set 2

Example: Construct a sketch of a graph of a function f such that f passes through the point $(4, 4)$, which is a relative minimum, and $f''(x) > 0$ everywhere.

Example: Construct a sketch of a graph of a function f such that

$f(0) = 0$

$f'(-3) = 0$ and $f'(4) = 0$

$f'(x) < 0$ on $(-3, 0)$ and $(0, 4)$

$f'(x) > 0$ on $(-\infty, -3)$ and $(4, \infty)$

$f''(x) < 0$ on $(-\infty, 0)$ and $f''(x) > 0$ on $(0, \infty)$

2^{nd} Derivative Test … Set 2

The Second Derivative Test

If a function $f(x)$ is such that both $f'(x)$ and $f''(x)$ can be computed, and c is a critical value of f,

1. If $f''(c) > 0$, then f has a relative minimum at $x = c$.
2. If $f''(c) < 0$, then f has a relative maximum at $x = c$.
3. If either $f''(c) = 0$ or $f''(c)$ is undefined, then the test fails and cannot be used to obtain information about the relative extrema of f at $x = c$, i.e. **go back to the first derivative test.**

2^{nd} Derivative Test … Set 2

Example: Summarize the behavior of $f(x) = -2x^3 + 3x^2 + 12x$ and sketch its graph.

2^{nd} Derivative Test … Set 2

Point of Diminishing Returns

When producing items, the **point of diminishing returns** is the point when increasing the input variable causes the marginal change per unit of output to start to decrease, i.e. it takes production of more items to cause the same increase as previously.

*This is the point at which the concavity of the function changes from concave up to concave down!

2^{nd} Derivative Test … Set 2

Example: An efficiency study conducted on the day shift of a manufacturer showed that the number of units of a product produced by a typical employee t hours after 8:00 AM could be modeled by the function

$$f(x) = -\frac{4}{3}t^3 + 8t^2 + 22t, \ 0 \leq t \leq 8$$

Find the point of diminishing returns for this model.

2^{nd} Derivative Test … Set 2

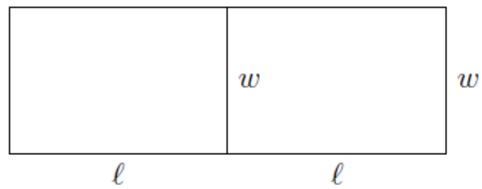
Applications of the Derivative: Optimization

Definition [Optimization]: The process of determining the maxima or minima values of a function $f(x)$ is called **optimization**.

Example: A homeowner wishes to insulate his attic with insulation that is r inches thick. To cover the entire attic, it will cost \$100 per inch of insulation. He also predicts that the heating costs over the next 10 years will be about \$3,000 if he uses 1 inch of insulation and will decrease proportionally to the number of inches of insulation added. How many inches of insulation should be placed in the attic if the total cost of insulation is to be minimized?

2^{nd} Derivative Test … Set 2

Example: A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals, as shown below. To enclose the maximum area, what should be the dimensions of each corral?



Example: A charter flight club charges its members \$250 per year. But for each new member above a membership of 65, the charge for all members is reduced by \$2 each. What number of members leads to a maximum revenue?

Example: A motel finds that it can rent 250 rooms per day if it charges \$120 per room. For each \$5 increase in rental rate, 8 fewer rooms will be rented per day. What room rate maximizes revenue?

2nd Derivative Test … Set 2

Applications of the Derivative in Business

Question: How many times a year should you re-order items for a warehouse to minimize costs?

Definition [Inventory Cost]: The **lot size**, x, is the number of units in an order.

The **holding/carrying cost**, $H(x)$, is the cost of holding onto x items before it can be sold.

$$\text{Holding Costs} = (\text{holding cost per unit}) \cdot (\text{average no. of units})$$

$$H(x) = (\text{holding cost per unit}) \cdot \frac{x}{2}$$

The **reordering cost**, $R(x)$, is the cost of ordering x items.

$$\text{Reordering Costs} = (\text{cost per order}) \cdot \left(\frac{\text{no. of units sold during time period}}{\text{lot size}}\right)$$

$$R(x) = (\text{cost per order}) \cdot \left(\frac{\text{no. of units sold during time period}}{x}\right)$$

The **total inventory cost**, $C(x)$, is given by

$$C(x) = H(x) + R(x)$$

Example: A company anticipates selling 2,400 units of its product at a uniform rate over the next year. Each time the company places an order for x units, it is charged a flat fee of \$50. Carrying costs are \$40 per unit per year. How many times should the company reorder each year and what should be the lot size of each order to minimize inventory cost? What is the minimum inventory cost?

2^{nd} Derivative Test … Set 2

Definition [The Economic Order Quanity (EOQ)]: The **economic order quantity (EOQ)** is the order quantity that minimizes total holding and ordering costs for the year. This will be the critical value of $C(x)$, and is given below:

$$x = \sqrt{\frac{2 \cdot (\text{cost per order}) \cdot (\text{units sold during time period})}{(\text{carrying cost per unit})}}$$

Example: Use the EOQ formula to compute the lot size that will minimize inventory cost for the company in the previous example.

Definition [Elasticity]: When a change in one variable causes a response in another variable, **elasticity** is a measure of the size of the response.

If a percent change in the input variable causes a *large* change in the output variable, the variables are said to be **elastic**. If the change is *small*, they are said to be **inelastic**.

When the variables involved are demand and price, we call this **elasticity of demand**, denoted η, and is given by

$$\eta = \frac{\frac{\Delta q}{q}}{\frac{\Delta p}{p}} = \frac{p}{q} \cdot \frac{\Delta q}{\Delta p}$$

dividing the percent change $(\frac{\Delta q}{q})$ in the quantity demanded (q) by the percent change $(\frac{\Delta p}{p})$ in the price (p) per unit.

2^{nd} Derivative Test … Set 2

Definition [Point Elasticity of Demand]: If the price p is a continuous function of the demand q, then

$$\lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} = \frac{dq}{dp}$$

Then, the **point elasticity of demand**, denoted ϵ, is the instantaneous rate of change of the η (times -1, since $\frac{dq}{dp}$ is almost always negative) shown below:

$$\epsilon = -\frac{p}{q} \cdot \frac{dq}{dp}$$

This gives the percent change in the demand for a 1% change in the price.

Example: A company estimates that the weekly sales q of its product is related to the product's price p by the function

$$q = \frac{4,000}{p^{3/2}} \tag{0.1}$$

where p is in dollars. Currently, each unit of the product is selling for \$25. Determine the point elasticity of demand of this product.

2nd Derivative Test … Set 2

Definition [Elastic, Inelastic, and Unit Elastic Demand]: Let

$$R = R(p) = q \cdot p = q(p) \cdot p$$

$$\text{so } R'(p) = q(1 - \epsilon)$$

denote revenue and ϵ denote the elasticity point of demand.

1. If $\epsilon > 1$ then $R'(p) < 0$, meaning the total revenue falls with price increases and rises with price decreases. Demand is **elastic**.
2. If $\epsilon = 1$ then $R'(p) = 0$, meaning the total revenue is unaffected by changes in price. Demand is **unit elastic**.
3. If $\epsilon < 1$ then $R'(p) > 0$, meaning the total revenue rises with price increases and falls with price decreases. Demand is **inelastic**.

Example: Suppose that a company is currently selling 50,00 units of a product at \$36 per unit. Suppose also that the point elasticity of demand for this product is $\epsilon = 0.41$. By how much, if any, does the revenue increase or decrease if the price is increased by 1%?