Maximizing and Output … Set 1

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Optimization

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Economics

The quantities we are usually concerned with in economic problems are demand (or price) $p(x)$, revenue $\mathcal{R}(x)$, cost $\mathcal{C}(x)$, and profit $\mathcal{P}(x)$.

These quantities are defined as follows:

- $p(x)$ is the price at which x units will be sold.
- $\mathcal{R}(x)$ is the revenue ("money in") earned from selling x units; hence,
$$\mathcal{R}(x) = x \cdot p(x).$$
- $\mathcal{C}(x)$ is the total cost ("money out") of producing x units.
- $\mathcal{P}(x)$ is the net profit (money in minus money out) of producing and selling x units; hence
$$\mathcal{P}(x) = \mathcal{R}(x) - \mathcal{C}(x).$$

Typically, but not always,
profit is maximized at a critical number for $\mathcal{P}(x)$, i.e., where

$$\mathcal{P}'(x) = \mathcal{R}'(x) - \mathcal{C}'(x) = 0,$$

or where marginal revenue $\mathcal{R}'(x)$ equals marginal cost $\mathcal{C}'(x)$

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Example 1

Example 1

A manufacturer determines that between 0 and 1000 units can be made, and that the total cost of producing x units is $\mathcal{C}(x) = -.01x^2 + 30x + 200$, while the revenue will be $\mathcal{R}(x) = 23x$. Find the profit function $\mathcal{P}(x)$ and determine the production level that will maximize profit.

The profit function has domain $[0, 1000]$ and is

$$\begin{aligned} \mathcal{P}(x) &= \mathcal{R}(x) - \mathcal{C}(x) \\ &= 23x - (-.01x^2 + 30x + 200) \\ &= .01x^2 - 7x - 200. \end{aligned}$$

Since $\mathcal{P}(x)$ is a continuous function on a closed interval,
its maximum value occurs either at a critical number or at an endpoint.

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Example 1

We have $\mathcal{P}(x) = .01x^2 - 7x - 200$, hence $\mathcal{P}'(x) = .02x - 7$, and the only critical number is $x = 7/.02 = 350$.

Observe that

$$\begin{aligned}
\mathcal{P}(0) &= .01(0^2) - 7(0) - 200 = -200 \\
\mathcal{P}(350) &= .01(350^2) - 7(350) - 200 = -1425 \\
\mathcal{P}(1000) &= .01(1000^2) - 7(1000) - 200 = 2800,
\end{aligned}$$

and so profit is maximized at a production level of $\boxed{x = 1000}$ units.

Note that the maximum value does not occur at the critical number. In fact, the absolute *minimum* profit occurs at the critical number $x = 350$.

its maximum value occurs either at a critical number or at an endpoint.

Example 2

Example 2

A manufacturer determines that between 0 and 100 units can be made and sold each week.
The total cost of producing x units is $\mathcal{C}(x) = .01x^3 - 1.2x^2 + 50x + 1000$.
The price that can be charged for each unit is $p(x) = \$50$.
Determine the revenue function $\mathcal{R}(x)$ and the profit function $\mathcal{P}(x)$, and determine the production level that will maximize profit.

First, $\mathcal{R}(x) = x \cdot p(x) = x \cdot 50$, and so $\mathcal{R}(x) = 50x$.
Therefore, $\mathcal{P}(x)$ has domain $[0, 100]$ and is

$$\begin{aligned} \mathcal{P}(x) &= \mathcal{R}(x) - \mathcal{C}(x) \\ &= 50x - (.01x^3 - 1.2x^2 + 50x + 1000) \\ &= -.01x^3 + 1.2x^2 - 1000, \end{aligned}$$

Again, $\mathcal{P}(x)$ is a continuous function on a closed interval, and so its maximum value occurs at a critical number or at an endpoint.

Example 2

We have $\mathcal{P}(x) = -.01x^3 + 1.2x^2 - 1000$, hence

$$\mathcal{P}'(x) = -.03x^2 + 2.4x = -.03x(x - 80),$$

and the critical numbers are $x = 0$ and $x = 80$.

Observe that

$$\begin{aligned}
\mathcal{P}(0) &= -.01(0^3) + 1.2(0^2) - 1000 = -1000 \\
\mathcal{P}(80) &= -.01(80^3) + 1.2(80^2) - 1000 = 1560 \\
\mathcal{P}(100) &= -.01(100^3) + 1.2(100^2) - 1000 = 1000,
\end{aligned}$$

and so profit is maximized at a production level of $\boxed{x = 80}$ units.

Example 3

Example 3

A theater seats 1000 people, and if the ticket price is $15, then the play will sell out. For each increase of $1 in the ticket price, 50 less tickets will be sold. What price should be charged in order to maximize the revenue from ticket sales?

The revenue from ticket sales is the product of the number of tickets sold and the ticket price. We let

- x = number of $1 increases in the ticket price.

We then have

- ticket price $= 15 + x$ dollars,
- number of tickets sold $= 1000 - 50x$.

We want to maximize revenue

$$R(x) = (15 + x)(1000 - 50x) = 15000 + 250x - 50x^2$$

on the interval $[0, \infty)$.

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Example 3

We want to find the absolute maximum value of
$R(x) = 15000 + 250x - 50x^2$ on $[0, \infty)$.
We need the derivative in order to find any critical numbers of $R(x)$:

$$R'(x) = 250 - 100x.$$

The only critical number of $R(x)$ in $[0, \infty)$ is where $250 - 100x = 0$, hence $x = 2.5$.

Verify that $R(x)$ has an absolute maximum at $x = 2.5$:

- $R''(x) = -100$, so $R''(2.5) = -100$ is negative.
  By the Second Derivative Test, R has a *local* maximum at $x = 2.5$.
- The domain of R is an interval and R has only one critical number, hence R has an *absolute* maximum at $x = 2.5$.

Therefore, the ticket price that maximizes revenue is $15 + 2.5$, or $\boxed{\$17.50.}$