Differentiation Quiz … Set 7

Part I: Multiple Choice: No Calculator

1) If $f(x) = \frac{x^2-9}{x+3}$ has a removable discontinuity at $x = -3$, then $f(-3) =$

A) 3
B) -3
C) 0
D) 6
E) -6

2) $\lim_{x\to\infty} \frac{10^8x^5 + 10^6x^4 + 10^4x^2}{10^9x^6 + 10^7x^5 + 10^5x^3}$

A) 0
B) 1
C) -1
D) $\frac{1}{10}$
D) $-\frac{1}{10}$

3) If $f(x) = \sqrt{4\sin 2x + 2}$, then $f'(0) =$

A) -2
B) 0
C) $\sqrt{2}$
D) $2\sqrt{2}$
E) 1

Differentiation Quiz … Set 7

Answers

Part I: Multiple Choice: No Calculator

1) If $f(x) = \frac{x^2-9}{x+3}$ has a removable discontinuity at $x = -3$, then $f(-3) =$

A) 3
B) -3
C) 0
D) 6
E) -6

This function has a discontinuity where the denominator is zero. i.e., at $x = -3$. If this discontinuity is removed, then, we have:

$$f(x) = \frac{x^2 - 9}{x + 3} = x - 3$$

So, $f(-3) = -3 - 3 = -6$

Answer: E

2) $\lim_{x\to\infty} \frac{10^8x^5 + 10^6x^4 + 10^4x^2}{10^9x^6 + 10^7x^5 + 10^5x^3}$

A) 0
B) 1
C) -1
D) $\frac{1}{10}$
D) $-\frac{1}{10}$

For this rational function, the highest power of x takes over when calculating a limit. Since the x^6 term exists in the denominator and not in the numerator, the ratio of the two polynomials gets smaller and smaller as $x \to 0$, eventually approaching 0.

Alternatively, apply L'Hospital's rule repeatedly (5 times) until you have a constant in the numerator and a linear term in the denominator. The ratio can then be seen to approach zero as $x \to \infty$.

Answer: A

3) If $f(x) = \sqrt{4\sin 2x + 2}$, then $f'(0) =$

A) -2
B) 0
C) $\sqrt{2}$
D) $2\sqrt{2}$
E) 1

$$f(x) = (4\sin 2x + 2)^{1/2}$$

$$f'(x) = \frac{1}{2}(4\sin 2x + 2)^{-1/2} \cdot (4\cos 2x) \cdot 2$$

$$= \frac{(4\cos 2x)}{\sqrt{4\sin 2x + 2}}$$

$$f'(0) = \frac{(4\cos 0)}{\sqrt{4\sin 0 + 2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

Answer: D

4) Let f and g be differentiable functions such that

$$f(1) = 4,\ g(1) = 3,\ f'(3) = -5$$

$$f'(1) = -4,\ g'(1) = -3,\ g'(3) = 2$$

If $h(x) = f(g(x))$, then $h'(1) =$

A) -9
B) 15
C) 0
D) -5
E) -12

5) What is $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)-\cos\left(\frac{\pi}{2}\right)}{h}$?

(A) -1
B) $-\frac{\sqrt{2}}{2}$
C) 0
D) 1
E) The limit does not exist.

6) If $f(x) = \sin^2(3 - x)$, then $f'(0) =$

A) $-2\cos 3$
B) $-2\sin 3\cos 3$
C) $6\cos 3$
D) $2\sin 3\cos 3$
E) $6\sin 3\cos 3$

Differentiation Quiz … Set 7

Answers

4) Let f and g be differentiable functions such that

$$f(1) = 4,\ g(1) = 3,\ f'(3) = -5$$

$$f'(1) = -4,\ g'(1) = -3,\ g'(3) = 2$$

If $h(x) = f(g(x))$, then $h'(1) =$

A) -9
B) 15
C) 0
D) -5
E) -12

Use the chain rule in Lagrange (Prime) Notation.

$$h'(x) = f'\big(g(x)\big) \cdot g'(x), \qquad \text{where: } h = f \circ g$$

$$h'(1) = f'\big(g(1)\big) \cdot g'(1) = f'(3) \cdot g'(1) = (-5) \cdot (-3) = 15$$

Answer: B

5) What is $\lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)-\cos\left(\frac{\pi}{2}\right)}{h}$?

A) -1
B) $-\frac{\sqrt{2}}{2}$
C) 0
D) 1
E) The limit does not exist.

This limit is the definition of a derivative.

$$\lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2}+h\right) - \cos\left(\frac{\pi}{2}\right)}{h} = \frac{d}{dx}(\cos x)\Big|_{x = \frac{\pi}{2}}$$

$$= (-\sin x)\Big|_{x=\frac{\pi}{2}} = -\sin\frac{\pi}{2} = -1$$

Answer: A

6) If $f(x) = \sin^2(3 - x)$, then $f'(0) =$

(A) $-2\cos 3$
(B) $-2\sin 3\cos 3$
(C) $6\cos 3$
(D) $2\sin 3\cos 3$
(E) $6\sin 3\cos 3$

$$f(x) = \sin^2(3 - x)$$

$$f'(x) = 2 \cdot \sin(3 - x) \cdot \frac{d}{dx}[\sin(3 - x)]$$

$$= 2 \cdot \sin(3 - x) \cdot \cos(3 - x) \cdot \frac{d}{dx}(3 - x)$$

$$= 2 \cdot \sin(3 - x) \cdot \cos(3 - x) \cdot (-1)$$

$$f'(0) = -2 \cdot \sin(3) \cdot \cos(3)$$

Answer: B

7) If $\lim_{x \to 2} \frac{f(x)}{x-2} = f'(2) = 0$, which of the following must be true?

I. $f(2) = 0$
II. $f(x)$ is continuous at $x = 2$
III. $f(x)$ has a horizontal tangent line at $x = 2$

A) I only
B) II only
C) I and II only
D) II and III only
E) I, II, and III

8) If $f(x) = x - 1$ and $g(x) = x^2 + 1$, then $f(g(x)) = g(f(x))$ when $x =$

A) $-\frac{1}{2}$
B) $\frac{1}{2}$
C) -1
D) 1
E) 0

Answers

7) If $\lim_{x\to 2}\frac{f(x)}{x-2} = f'(2) = 0$, which of the following must be true?

I. $f(2) = 0$
II. $f(x)$ is continuous at $x = 2$
III. $f(x)$ has a horizontal tangent line at $x = 2$

A) I only
B) II only
C) I and II only
D) II and III only
E) I, II, and III

Check the conditions one at a time.

I. $f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$

if, $f'(2) = \lim_{x\to 2}\frac{f(x)}{x-2} = \lim_{x\to 2}\frac{f(x)-0}{x-2}$

it must be true that $f(2) = 0$ **I is TRUE**

II. f is differentiable at $x = 2$, and so must be continuous at $x = 2$.
II is TRUE

III. The first derivative provides the slope of the tangent line. Since $f'(2) = 0$, there must be a tangent of slope 0 (horizontal) at $x = 2$

III is TRUE

Answer: E

8) If $f(x) = x - 1$ and $g(x) = x^2 + 1$, then $f(g(x)) = g(f(x))$ when $x =$

A) $-\frac{1}{2}$
B) $\frac{1}{2}$
C) -1
D) 1
E) 0

$f(g(x)) = f(x^2+1) = x^2+1-1 = x^2$

$g(f(x)) = g(x-1) = (x-1)^2+1 = x^2-2x+1+1 = x^2-2x+2$

Then, set: $f(g(x)) = g(f(x))$

$x^2 = x^2 - 2x + 2$

$2x = 2$

$x = 1$

Answer: D

9) Which of the following are equal to $\cos(2x)$?

I. $\cos^2 x - \sin^2 x$
II. $\cos^2 x + \sin^2 x$
III. $2\cos^2 x - 1$

A) I only
B) II only
C) I and III
D) all
E) none

10) What are the asymptotes of $f(x) = \frac{(x-14)^2}{(x+18)(x-14)}$?

A) horizontal at $y = 0$, no vertical
B) horizontal at $y = 0$, vertical at $x = -18$
C) horizontal at $y = 0$, vertical at $x = -18$ and $x = 14$
D) horizontal at $y = 1$, vertical at $x = -18$
E) horizontal at $y = 1$, vertical at $x = -18$ and $x = 14$

Differentiation Quiz … Set 7

Answers

9) Which of the following are equal to $\cos(2x)$?

I. $\cos^2 x - \sin^2 x$
II. $\cos^2 x + \sin^2 x$
III. $2\cos^2 x - 1$

A) I only
B) II only
C) I and III
D) all
E) none

Here are the trigonometry double-angle formulas:

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1 - tan^2 A}$$

Answer: C

10) What are the asymptotes of $f(x) = \dfrac{(x-14)^2}{(x+18)(x-14)}$?

A) horizontal at $y = 0$, no vertical
B) horizontal at $y = 0$, vertical at $x = -18$
C) horizontal at $y = 0$, vertical at $x = -18$ and $x = 14$
D) horizontal at $y = 1$, vertical at $x = -18$
E) horizontal at $y = 1$, vertical at $x = -18$ and $x = 14$

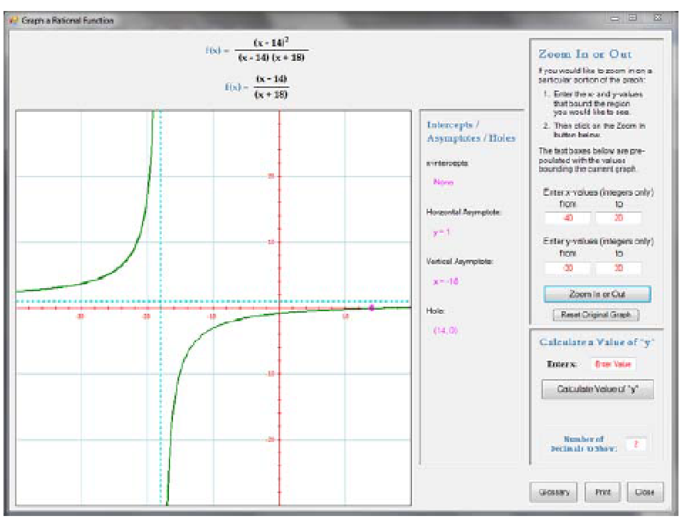

Image from the Algebra App available at www.mathguy.us

This is a rational function with a hole at $x = 14$ and a vertical asymptote at $x = -18$.

- A **hole** exists if the multiplicity of a root in the denominator $\leq$ the multiplicity of the same root in the numerator.
- A **vertical asymptote** exists if the multiplicity of a root in the denominator $>$ the multiplicity of the same root in the numerator.

To get the horizontal asymptote, calculate:

$$\lim_{x\to\infty} \frac{(x-14)^2}{(x+18)(x-14)} =$$

$$\lim_{x\to\infty} \frac{x^2 - 28x + 196}{x^2 + 4x - 252}$$

Using L'Hospital's Rule, take the derivatives of the numerator and denominator twice, to get $f(x) = 1$.

Answer: D

Part II: Multiple Choice: Calculators Are Allowed

11) The equation of the tangent line to the curve $x^2 + y^2 = 169$ at the point (5, -12) is

A) $5y - 12x = -120$
B) $5x - 12y = 119$
C) $5x - 12y = 169$
D) $12x + 5y = 0$
E) $12x + 5y = 169$

Differentiation Quiz … Set 7

Answers

Part II: Multiple Choice: Calculators Are Allowed

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A) $5y - 12x = -120$
B) $5x - 12y = 119$
C) $5x - 12y = 169$
D) $12x + 5y = 0$
E) $12x + 5y = 169$

Use implicit differentiation:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(169)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{5}{-12} = \frac{5}{12}$$

So, the slope of the tangent line, $m = \frac{5}{12}$

Then, using the point-slope form of a line:

$$y + 12 = \frac{5}{12}(x - 5)$$

$$12y + 144 = 5x - 25$$

$$169 = 5x - 12y$$

Answer: C

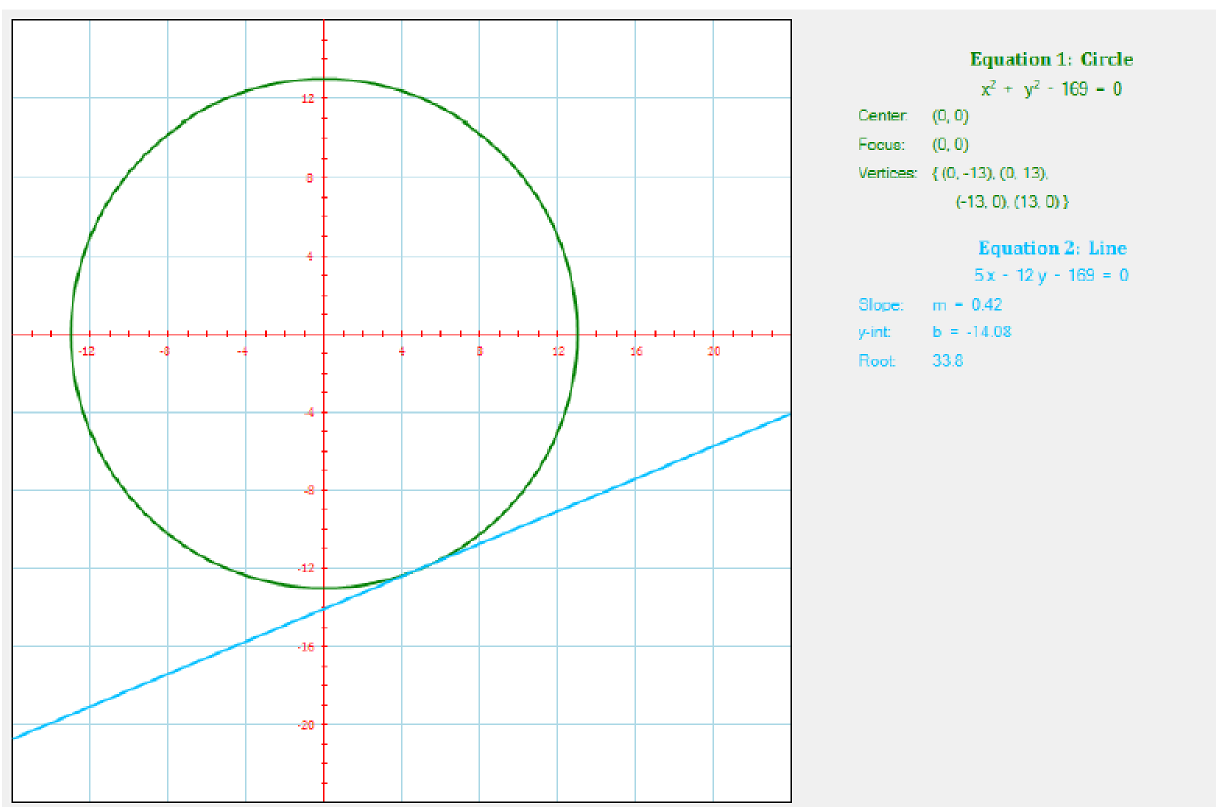

Image from the Algebra App available at www.mathguy.us

12) A point moves along the curve $y = x^2 + 1$ in such a way that when $x = 4$, the x-coordinate is increasing at the rate of 5 ft/sec. At what rate is the y-coordinate changing at that time?

A) 80 ft/sec
B) 45 ft/sec
C) 32 ft/sec
D) 85 ft/sec
E) 40 ft/sec

13) The equation of the line tangent to the curve $y = \frac{kx + 8}{k + x}$ at $x = -2$ is $y = x + 4$. What is the value of k?

A) -3
B) -1
C) 1
D) 3
E) 4

Differentiation Quiz … Set 7

Answers

12) A point moves along the curve $y = x^2 + 1$ in such a way that when $x = 4$, the x-coordinate is increasing at the rate of 5 ft/sec. At what rate is the y-coordinate changing at that time?

A) 80 ft/sec
B) 45 ft/sec
C) 32 ft/sec
D) 85 ft/sec
E) 40 ft/sec

This is a related rates problem with: $\frac{dx}{dt} = 5$ when $x = 4$.

$$\frac{d}{dt}(y) = \frac{d}{dt}(x^2 + 1)$$

$$\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$$

Substitute values for $x = 4$ and $\frac{dx}{dt} = 5$ to get:

$$\frac{dy}{dt} = 2 \cdot (4) \cdot (5) = 40 \text{ ft/sec}$$

Answer: E

13) The equation of the line tangent to the curve $y = \frac{kx + 8}{k + x}$ at $x = -2$ is $y = x + 4$. What is the value of k?

A) -3
B) -1
C) 1
D) 3
E) 4

Use either the product or quotient rule to calculate $\frac{dy}{dx}$. Note also that $\frac{dy}{dx} = 1$ because the slope of the line $y = x + 4$ is 1.

$$\frac{dy}{dx} = \frac{(k + x) \cdot \frac{d}{dx}(kx + 8) - (kx + 8) \cdot \frac{d}{dx}(k + x)}{(k + x)^2}$$

$$1 = \frac{(k + x) \cdot (k) - (kx + 8) \cdot (1)}{(k + x)^2} = \frac{(k - 2) \cdot (k) - (-2k + 8) \cdot (1)}{(k - 2)^2}$$

substituting -2 for the value of x

$$(k - 2)^2 = k^2 - 2k + 2k - 8$$

$$k^2 - 4k + 4 = k^2 - 8$$

$$-4k = -12$$

$$k = 3$$

Answer: D

Differentiation Quiz … Set 7

Part III: Free Response: No Calculators

1) Let f be the function given by $f(x) = \frac{x}{\sqrt{x^2-4}}$
   a) Find the domain of f. (Write your answer in interval notation.)
   b) Write an equation for each vertical asymptote to the graph of f.
   c) Write an equation for each horizontal asymptote to the graph of f.
   d) Find $f'(x)$.

Differentiation Quiz … Set 7

Answers

Part III: Free Response: No Calculators

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   a) Find the domain of f. (Write your answer in interval notation.)
   b) Write an equation for each vertical asymptote to the graph of f.
   c) Write an equation for each horizontal asymptote to the graph of f.
   d) Find $f'(x)$.

a. The domain of x exists where the denominator is real and non-zero.

$$x^2 - 4 > 0 \quad \Rightarrow \quad x^2 > 4 \quad \Rightarrow \quad |x| > 2$$

b. Vertical asymptotes exist where the denominator is zero.

$$x^2 - 4 = 0 \quad \Rightarrow \quad |x| = 2 \quad \Rightarrow \quad x = \pm 2$$

c. Left horizontal asymptote (easy method is to eliminate the constant):

$$y = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2-4}} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2}} = \lim_{x\to-\infty} \frac{x}{|x|} = -1 \qquad \Rightarrow \qquad y = -1$$

Right horizontal asymptote (easy method is to eliminate the constant):

$$y = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2-4}} = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2}} = \lim_{x\to+\infty} \frac{x}{|x|} = 1 \qquad \Rightarrow \qquad y = 1$$

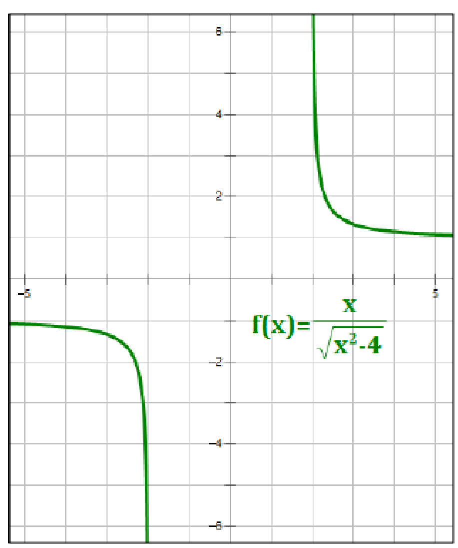
d. $$f'(x) = \frac{(x^2-4)^{1/2} \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}\left[(x^2-4)^{1/2}\right]}{x^2-4}$$

$$= \frac{(x^2-4)^{1/2} \cdot 1 \; - \; x \cdot \frac{1}{2}\left[(x^2-4)^{-1/2}\right] \cdot 2x}{x^2-4}$$

$$= \frac{(x^2-4)^{1/2} \; - \; x^2 \cdot \left[(x^2-4)^{-1/2}\right]}{x^2-4}$$

$$= \frac{(x^2-4)^{-1/2} \cdot [(x^2-4)-x^2]}{x^2-4}$$

$$= \frac{-4}{(x^2-4)^{1/2} \cdot (x^2-4)} = \frac{-4}{(x^2-4)^{3/2}}$$



Differentiation Quiz … Set 7

2) Given $f(x) = 8x^3$, write an equation for the line tangent to the graph of f at $x = \frac{1}{2}$.

Differentiation Quiz … Set 7

Answers

2) Given $f(x) = 8x^3$, write an equation for the line tangent to the graph of f at $x = \frac{1}{2}$.

The slope of the tangent line is equal to the derivative of the function at that point.

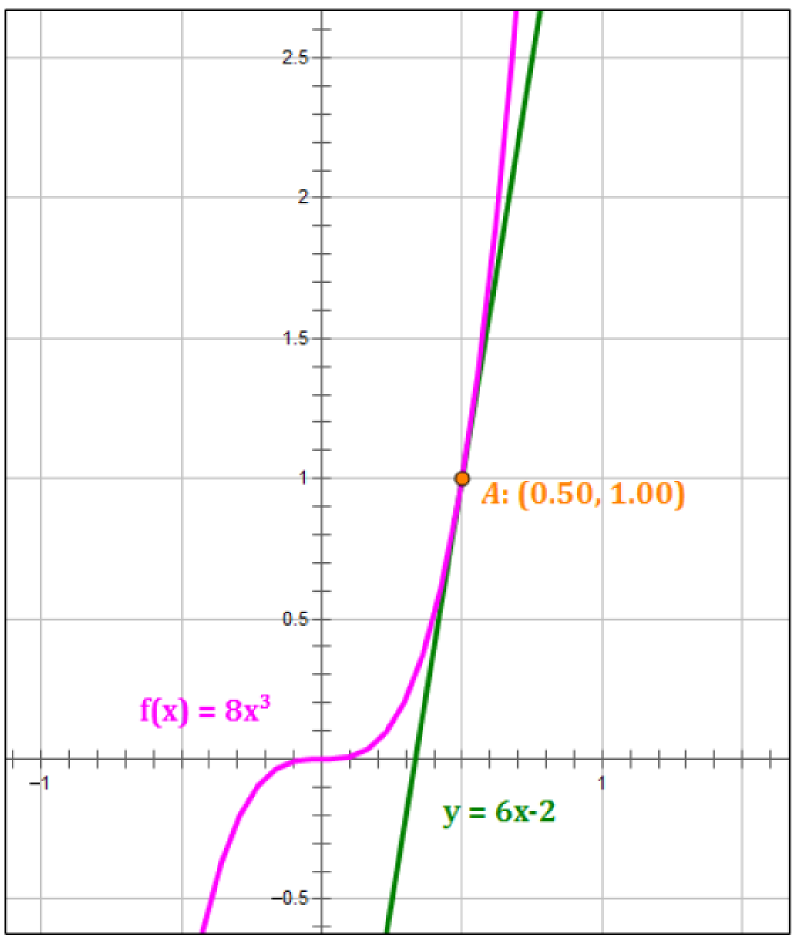
$$\frac{d}{dx}(f(x)) = 24x^2 \qquad \therefore \qquad m = 24 \cdot \left(\frac{1}{2}\right)^2 = 6$$

Next, find $f\left(\frac{1}{2}\right) = 8 \cdot \left(\frac{1}{2}\right)^3 = 1$

So, $\left(\frac{1}{2}, 1\right)$ is a point on the curve.

Then, in point-slope form, the equation is: $y - 1 = 6\left(x - \frac{1}{2}\right)$

In slope-intercept form, this is: $y = 6x - 2$



Part IV: Free Response: Calculator Portion

3) Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \leq 0 \\ 3\cos x - 2 & \text{for } x > 0. \end{cases}$

a) Show that f is continuous at $x = 0$.

b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function.

Answers

Part IV: Free Response: Calculator Portion

3) Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ 3\cos x - 2 & \text{for } x > 0. \end{cases}$

Both "piece functions" are continuous.

a) Show that f is continuous at $x = 0$.

b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function.

a. The following items are required for a piecewise function to be continuous:

- Each piece must be continuous
- The function must exist at points where the pieces "come together."
- The limits from the left and right must both equal the value of the curve at points where the pieces "come together."

Now, consider f :

Note that each of the two component functions of f is continuous. So, we need only check the point at which the two curves "come together", i.e., $x = 0$.

We require that $f(0)$ exists. It does exist and, from the top function, $f(0) = 1$.

Finally, check the limits at $x = 0$ from the left and the right:

$$\lim_{x \to 0-} (1 - 2\sin x) = 1 - 2 \cdot 0 = 1$$

$$\lim_{x \to 0+} (3\cos x - 2) = 3 \cdot 1 - 2 = 1$$

So, we see that: $\lim_{x \to 0-} (f(x)) = \lim_{x \to 0+} (f(x)) = f(0)$

Therefore, the function is continuous.

b. Take the derivative of each piece. Note that we are given $x \neq 0$.

$$\frac{d}{dx}(1 - 2\sin x) = -2\cos x \qquad \text{for } \; x < 0$$

$$\frac{d}{dx}(3\cos x - 2) = -3\sin x \qquad \text{for } \; x > 0$$

4) At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill, and this model is estimated to work for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010. $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

b) Find $\frac{d^2W}{dt^2}$ in terms of W.

Differentiation Quiz … Set 7

Answers

4) At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill, and this model is estimated to work for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010. $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

b) Find $\frac{d^2W}{dt^2}$ in terms of W.

A note on notation: W_a and $f(a)$ both refer to the value of the function at $x = a$.

a. **See the image at right. We are given:**

$$W_0 = f(0) = 1,400$$

$$\frac{dW}{dt} = \frac{1}{25}(W - 300)$$

Notice that the curve is close to a straight line on the interval we care about, so we can approximate the change from $t = 0$ to $t = 0.25$ as linear.

The slope of the curve at $t = 0$ is:

$$\left.\frac{dW}{dt}\right|_{W=1,400} = \frac{1}{25}(1,400 - 300)$$

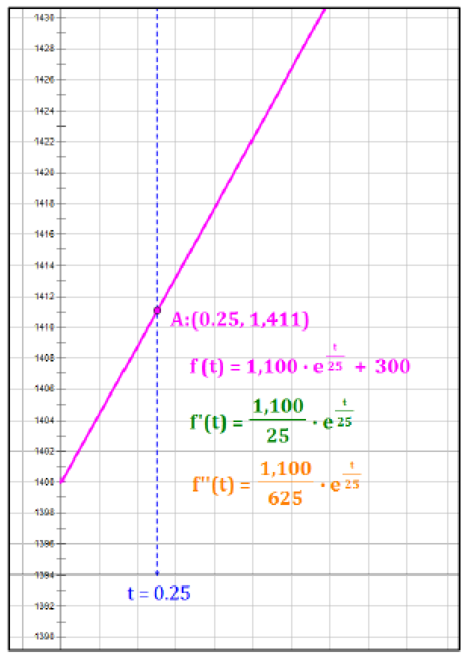
$$= 44 \text{ tons/year}$$

Estimate, then, that

$$W_1 = f(1) \sim 1,400 + 44 = 1,444 \text{ tons}$$

$$W_{0.25} = f(0.25) \sim 1,400 + (.25) \cdot (44)$$

$$\sim 1,411 \text{ tons}$$



The above curve, $f(t)$, can be derived using Integral Calculus, to which the student has not yet been exposed. Nevertheless, the illustration may be helpful in visualizing the situation.

b. $$\frac{d^2W}{dt^2} = \frac{d}{dt}\left(\frac{dW}{dt}\right) = \frac{d}{dt}\left[\frac{1}{25}(W - 300)\right]$$

$$= \frac{1}{25} \cdot \frac{dW}{dt} = \frac{1}{25} \cdot \left[\frac{1}{25}(W - 300)\right] = \frac{1}{625}(W - 300)$$