THE SECOND DERIVATIVE

Summary

The previous section allowed us to analyse a function by its first derivative. Static, critical, minimum and maximum points could all be determined with the first derivative.

Even if the first derivative gives a lot of information about a function, it does not characterize it completely.

Consider for example a function $f(x)$ with $f(0) = 0$ and $f(1) = 1$ and suppose that its first derivative $f'(x)$ is positive for all values of x in the interval $[0,1]$. Is the information regarding the function and its derivative sufficient to have a precise idea of the behaviour of the function?

The answer is no. All we can say is that the function passes by the points $(0,0)$ and $(1,1)$ and that it is increasing between the interval $[0,1]$. The following graph illustrates 3 functions that satisfy these conditions :
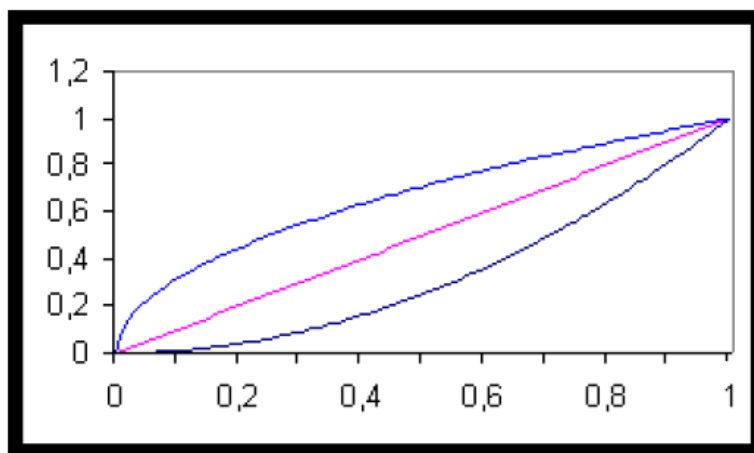
The three functions all pass through points $(0,0)$ and $(1,1)$ and are increasing between the given interval however there is a difference in the way each function gets to the next point. They differ by their curve.

1. Curvature- concavity and convexity

An intuitive definition: a function f is said to be **convex** at an interval I if, for all pairs of points on the $f(x)$ graph, the line segment that connects these two points passes above the $f(x)$ curve.

A function f is said to be concave at an interval I if, for all pairs of points on the $f(x)$ graph, the line segment that connects these two points passes below the $f(x)$ curve.

Theorem :

Consider $f(x)$, a function that is twice continuously differentiable on an interval I. The function f is:

- convex if $f''(x) > 0$ for all x in I
- concave if $f''(x) < 0$ for all x in I

Interpretation

A convex function has an increasing first derivative, making it appear to bend upwards. Contrarily, a concave function has a decreasing first derivative making it bend downwards. It is important to understand the distinction between the first derivative, which informs us of the slope of the tangent of a function, and the second derivative, which shows us how it is curved.
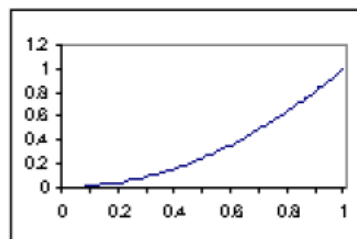
Is an increasing function always convex? Is a decreasing function always concave? Justify your answer with the help of the following diagrams.
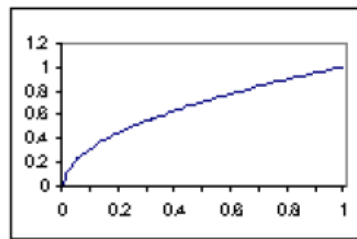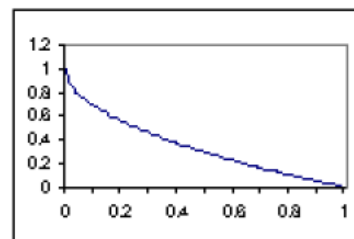

f' >0, f" <0
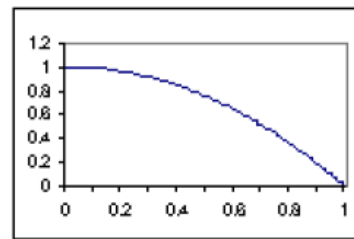
f' <0, f" <0

f' >0, f" >0

f' <0, f" >0

Example 1

Examine the curve of the function $f(x) = x^2 + 2x - 4$.

Solution

$f(x) = x^2 + 2x - 4$

$f'(x) = 2x + 2$

$f''(x) = 2 > 0$

The function $f(x) = x^2 + 2x - 4$ is such that $f''(x) > 0$ for all x. It is therefore always convex.

Example 2

Find the intervals for which the function $f(x) = \ln(x^2 + 1)$ is concave.

Solution

By definition, a function is concave if $f''(x) < 0$.

$$f'(x) = \frac{1}{x^2+1} \cdot (x^2 + 1)' \qquad \text{(chain rule)}$$

$$= \frac{2x}{x^2 + 1}$$

$$f''(x) = \frac{(2x)'.(x^2+1) - 2x.(x^2+1)'}{(x^2+1)^2} \qquad \text{(quotient rule)}$$

$$= \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2}$$

$$= \frac{2x^2 + 2 - 4x^2}{(x^2 + 1)^2}$$

$$= \frac{2 - 2x^2}{(x^2 + 1)^2}$$

Since the denominator of $f''(x)$ is always positive, the sign of $f''(x)$ will be determined by the numerator: $2 - 2x^2 = 2(1 - x^2) = 2(1 - x)(1 + x)$.

The best way to proceed is to find the values of x for which

$2(1 - x)(1 + x) = 0$ and examine the sign of $2(1 - x)(1 + x)$ between and around these values (somewhat like a table of variation).

$2(1 - x)(1 + x) = 0$ when $x = 1$ and $x = -1$

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
Sign of f''	-	0	+	0	-
Concave	concave ∩		convex ∪		concave ∩
Convex					

Since a function is concave when $f''(x) < 0$, we can conclude that $f(x) = \ln(x^2 + 1)$ is concave over the intervals $x < -1$ and $x > 1$. The following graph demonstrates the change in the curve of $f(x)$ at the points $x = -1$ et $x = 1$. (The inflection points are difficult to visualise precisely).
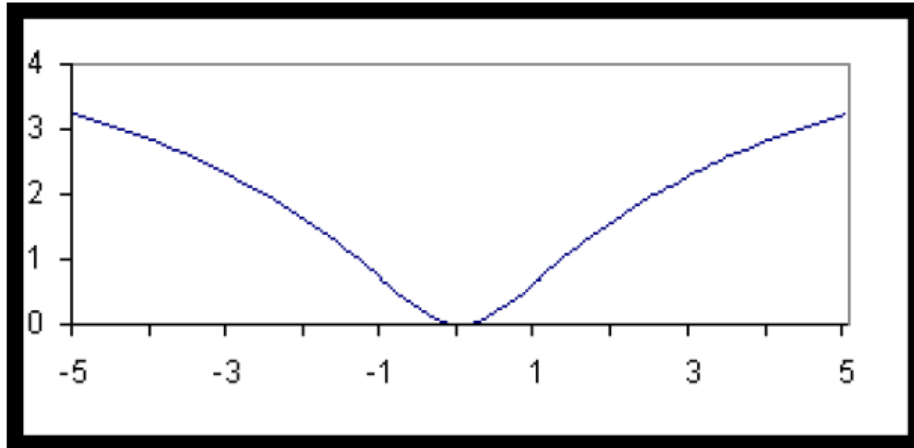
Exercise

1. Show that the function $f(x) = e^{x^2}$ never changes curvature.
2. Find interval for which the function is concave resp. convex.

$$f(x) = x^4 - 2x^3 - 12x^2 + 8x + 6.$$

Using Excel, plot the graph to confirm the results.

## 2. Determining the nature of a static point using the second derivative

We discussed the first derivative rule, that allowed us to verify if a static or critical point is a local optimum. The second derivative can also be used to determine the nature of a static point. However, the rule of the second derivative is limited to the study of static points.
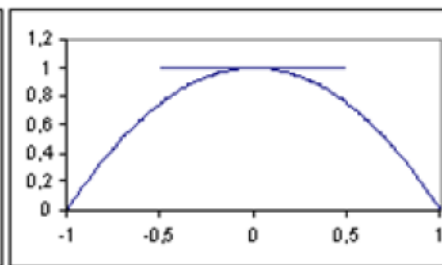
The second derivative rule

Given the function $f(x)$ and $x = x^*$ a static point of the function.

$f(x^*)$ is :

- a local minimum if the function is convex at x^* , i.e. $f''(x^*) > 0$;

- a local maximum if the function is concave at x^* , i.e. $f''(x^*) < 0$.
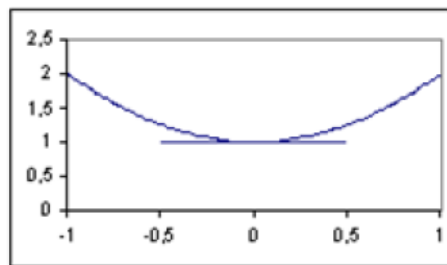
Convex function at a static point: local min

Concave function at a static point: local max

Essentially, the second derivative rule does not allow us to find information that was not already known by the first derivative rule. However, it may be faster and easier to use the second derivative rule.

Methodology : identification of the static points of $f(x)$ with the second derivative rule

1. Calculate the first derivative of $f(x)$;
2. Find all static points;
3. Calculate the second derivative of $f(x)$;
4. Evaluate $f''(x)$ at the static points;
5. Apply the second derivative rule. Conclude.

Example 1

Find all local optima of the function $f(x) = 2x^3 - 3x^2 - 12x + 4.$

1. **Calculate the first derivative of $f(x)$:**

   $f'(x) = 6x^2 - 6x - 12$

   $\qquad = 6(x^2 - x - 2)$

2. **Find all static points :**

   We obtain a static point when $f'(x) = 0.$

   $6(x^2 - x - 2) = 0$

   $x^2 - x - 2 = 0$

   $(x + 1)(x - 2) = 0 \rightarrow x = -1, x = 2$

   There are thus two static points $(x = -1, x = 2).$

3. **Calculate the second derivative :**

   $f''(x) = (6x^2 - 6x - 12)'$

   $\qquad = 12x - 6$

4. **Evaluate $f''(x)$ at the static points :**

   $f''(-1) = 12(-1) - 6$

   $\qquad = -18$

   $f''(2) = 12(2) - 6$

   $\qquad = 18$

5. **Apply the second derivative rule. Conclude :**

- At the static point $x = 1$, the second derivative $(f''(-1) = -18 < 0)$ is negative. The function is therefore concave at that point, indicating it is a local maximum.

- At the static point $x = 2$, the second derivative $(f''(2) = 18 > 0)$ is positive. The function is convex at that point indicating it is a local minimum.

A table of variations is therefore unnecessary when applying the second derivative rule. However, it is **essential** to remember that this rule may only be applied to determine the nature of static points, never the critical points of a function.

3. Global optima

The second derivative advantage is that it allows us to not only rapidly identify if a static point is a local optimum, but also if it is a global (or absolute) optimum:

Definition : *global maximum and minimum*

Given $f(x)$ a function defined on the domain $Dom(f)$. The point $x = x^*$ is

- an *global maximum* of f if and only if $f(x*) > f(x)$ for all $x \in Dom(f)$.

- an global *minimum* of f if and only if $f(x*) < f(x)$ for all $x \in Dom(f)$.

Identifying global optima using the second derivative :

Global maximum
Given $f(x)$ a function defined on the domain $Dom(f)$, and $x = x^*$, a static point of f. If the function is concave over its entire domain, i.e. if $f''(x) < 0$ for all $x \in Dom(f)$, then the function has an global maximum at point $x = x^*$.
Global minimum
Given $f(x)$ a function defined by the domain $Dom(f)$, and $x = x^*$, a static point of f. If the function is convex over its entire domain, i.e. if $f''(x) > 0$ for all $x \in Dom(f)$, then the function has an global minimum at point $x = x^*$.

Example

Find all optima of function $f(x) = x^4 + 8x^2 - 3$. Specify the type(s) of the optima.

Solution

$f(x) = x^4 + 8x^2 - 3$

$f'(x) = 4x^3 + 16x$

Static point $4x^3 + 16x = 0$

$$4x(x2 + 4) = 0$$

$$x = 0$$

$f''(x) = 12x^2 + 16 \rightarrow f''(0) = 16 > 0$

According to the second derivative rule, there is a local minimum at static point $x = 0$. Additionally, since the function is always convex ($f''(x) = 12x^2 + 16 > 0$), $f(0) = -3$ is an global (or absolute) minimum of f.

Exercise

Find the global minimum or maximum of each of the following functions.

a. $f(x) = e^{x^2}$, where $x \in R$

b. $\frac{x^2}{(x^2+1)}$, where $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

c. $\sqrt{x^2 + 1}$, where $x \in R$