Second Derivative

Business Calculus

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The Second Derivative Facts … Set 3

second derivative
definition of f'

- $f'(x)$ is the slope of the function $f(x)$
- $f'(x)$ is the rate of change of the function $f(x)$ w.r.t x
- $f'(x)$ is $(+)$ where $f(x)$ is increasing.
- $f'(x)$ is $(-)$ where $f(x)$ is decreasing.
- $f'(x)$ can be used to find intervals where $f(x)$ is increasing/decreasing
- $f'(x) = 0$ **or undefined** where the sign of the slope can change, i.e. at CNs
- $f'(x) = 0$ at critical points
- The **First Derivative Test** can be used to determine what kind of critical points.
- $f'(x)$ can tell you a lot about a function $f(x)$
- $f'_{avg} = \frac{\Delta f}{\Delta x}$ can be used to extimate $f'(x)$

Roy M. Lowman Second Derivative

second derivative

definition of f''

Definition (second derivative)

$$\frac{d^2f}{dx^2} = \frac{d}{dx}f'(x) = \lim_{\Delta x -> 0} \frac{f'(x + \Delta x) - f(x)'}{\Delta x} \quad (5)$$

$$= \lim_{\Delta x -> 0} \frac{\Delta f'}{\Delta x} \quad (6)$$

$$\approx \frac{\Delta f'}{\Delta x} \text{average slope of } f' \text{over} \Delta x, \quad f''_{avg} \quad (7)$$

$$(8)$$

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definition of f''

- $f''(x)$ gives the **concavity** of function $\mathbf{f(x)}$
- $f''(x)$ is the rate of change of slope w.r.t $\mathbf{x}$
- $f''(x)$ is $(+)$ where $\mathbf{f(x)}$ is concave up. (holds $\mathbf{H_2O}$)
- $f''(x)$ is $(-)$ where $\mathbf{f(x)}$ is concave down. (makes letter $\mathbf{A}$)
- $f''(x)$ can be used to find intervals where $\mathbf{f(x)}$ is **concave up** or **concave down**
- $f''(x)$ $=$ **0 or undefined** where the concavity can change
- $f''(x) = \mathbf{0}$ at **inflection points**, but must check if actually IP.
- The **Second Derivative Test** can be used to determine what type of critical points where $\mathbf{f'(x) = 0}$.
- $f''(x)$ can tell you a lot about a function $\mathbf{f(x)}$
- $f''_{avg} = \frac{\Delta f'}{\Delta x}$ can be used to extimate $f''(x)$

second derivative

missing slides TBA

TBA A few slides with notes from the lecture are missing here and will be added latter. The missing slides show how f“ is related to the curvature of a graph, inflection points and how the second derivative test works.

The Second Derivative Facts … Set 3

second derivative

example: find intervals concave up/down

Typical Exam Problem:

Given $f(x) = x^4 - 6x^2 - 12$, use f'' to determine where the graph of $f(x)$ is concave up and where it is concave down.

1st set $f'' = 0$ to determine where the concavity can change.

2nd evaluate $f''(x)$ at one test point in each interval.

- If $f'' = (+)$ (holds water) at one test point in an interval then $f(x)$ is concave up at that test point and at every point in the same interval.
- If $f'' = (-)$ (makes letter **A**) at one test point in an interval then $f(x)$ is concave down at that test point and at every point in the same interval.
- Repeat for one test point in each interval. Organize your work by drawing a number line with boundaries etc.

second derivative
example: find intervals concave up/down

Given $f(x) = x^4 - 6x^2 - 12$, use f'' to determine where the graph of $f(x)$ is concave up and where it is concave down.

1st set $f'' = 0$ to determine where the concavity can change.

- $f' = 4x^3 - 12x$
- $f'' = 12x^2 - 12 = 12(x^2 - 1) = 12(x-1)(x+1)$
- solve $f'' = 12(x-1)(x+1) = 0$ gives boundaries where concavity can change at $x = 1$, and $x = -1$

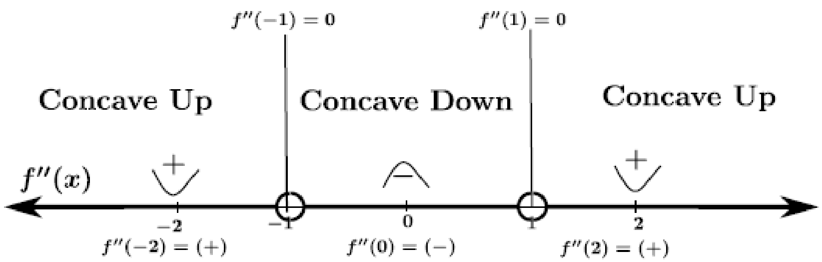
2nd evaluate $f''(x)$ at one test point in each interval.

- Convenient test points $x = -2$, 0 and 2
- $f''(-2) = 12((-2)-1)((-2)+1) = (-)(-) = (+)$ Positive (holds water) $\Rightarrow$ concave up in this interval.
- $f''(0) = 12((0)-1)((0)+1) = (-)(+) = (-)$ Negative (makes letter A) $\Rightarrow$ concave down in this interval.
- $f''(2) = 12((2)-1)((2)+1) = (+)(+) = (+)$ Positive (holds water) $\Rightarrow$ concave up in this interval.
- now organize results by drawing a number line with boundaries etc.

second derivative

example: find intervals concave up/down

Given $f(x) = x^4 - 6x^2 - 12$, use f'' to determine where the graph of $f(x)$ is concave up and where it is concave down.



$f(x)$ is concave up for $-\infty < x < -1$ and for $1 < x < \infty$
and
$f(x)$ is concave down for $-1 < x < 1$

The Second Derivative Facts … Set 3

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Second Derivative Test

- If $\mathbf{f'(x_c) = 0}$ then $\mathbf{x_c}$ is a critical number (**CN**)
- The point on the graph $\mathbf{(x_c, f(x_c))}$ is a critical point (**CP**).
- The second derivative can be used to determine what kind of **CP**.

Definition (Second Derivative Test)

- Given $\mathbf{x_c}$ is a critical number where $\mathbf{f'(x_c) = 0}$ then if:
- $f''(x_c) = (+)$ (holds water) then the **CP** is a **Relative Minimum**
- $f''(x_c) = (-)$ (makes letter A) then the **CP** is a **Relative Maximum**
- $\mathbf{f''(x_c) = 0}$ then the second derivative cannot determine what kind of CP and you must then use the **First Derivative Test**.

second derivative

example: second derivative text

Typical Exam Problem:

Given $f(x) = 100 - (x - 4)^2$, find all critical numbers x_c, find all critical points, then use the Second Derivative Test to determine what kind of CP

- $f' = -2(x - 4) = 2(4 - x) = 0$ gives $x_c = 4$
- Critical Point is $(4, 100)$
- $f''(x) = -2$
- now use the second derivative test
- $f''(4) = -2 = (-)$ Negative (makes letter A) indicates that the **CP** $(4, 100)$ is a relative maximum.

second derivative
inflection point

- An inflection point is a point where the concavity changes from up to down or down to up.
- An inflection point will occur where $f''(x) = 0$.
- However, the converse may not be true so you must check the sign of $f''(x)$ on each side of the point where $f''(x) = 0$.
- If the signs of $f''(x)$ are different on opposite sides of the point where $f''(x) = 0$ then the point is an inflection point.

second derivative

example: inflection point

Typical exam problem:
Given $\mathbf{f(x) = (x-2)^3 + 1}$, find all inflection points (if any).

- $\mathbf{f' = 3(x-2)^2}$
- $\mathbf{f'' = 6(x-2)}$
- solve $\mathbf{f'' = 6(x-2) = 0}$ for possible IPs.
- There may be an inflection at $\mathbf{x = 2}$
- The point $\mathbf{(2, 1)}$ may be an IP, need to check the sign of $\mathbf{f''(x)}$ on both sides of the point.
- $\mathbf{f''(1) = 6(1-2) = (-)}$ Negative (letter A) $\Rightarrow$ concave down on left of point
- $\mathbf{f''(3) = 6(3-2) = (+)}$ Positive (holds water) $\Rightarrow$ concave up on right of point
- since the concavity is different on opposite sides of the point, $\mathbf{(2, 1)}$ is an inflection point.