Local Maxima, Local Minima, and Inflection Points

Let f be a function defined on an interval $[a,b]$ or (a,b), and let p be a point in (a,b), i.e., not an endpoint, if the interval is closed.

- f has a ***local minimum*** at p if $f(p) \le f(x)$ for all x in a small interval around p.
- f has a ***local maximum*** at p if $f(p) \ge f(x)$ for all x in a small interval around p.
- f has an ***inflection point*** at p if the concavity of f changes at p, i.e. if f is concave down on one side of p and concave up on another.

We assume that $f'(p) = 0$ is only at isolated points — not everywhere on some interval. This makes things simpler, as then the three terms defined above are mutually exclusive.

The results in the tables below require that f is differentiable at p, and possibly in some small interval around p. Some of them require that f be twice differentiable.

Table 1: Information about f at p from the first and second derivatives at p

$f'(p)$	$f''(p)$	At p, f has a_____	Examples
0	positive	local minimum	$f(x) = x^2$, $p = 0$.
0	negative	local maximum	$f(x) = 1-x^2$, $p = 0$.
0	0	local minimum, local maximum, or inflection point	$f(x) = x^4$, $p = 0$. [min] $f(x) = 1-x^4$, $p = 0$. [max] $f(x) = x^3$, $p = 0$. [inf pt]
nonzero	0	possible inflection point	$f(x) = tan(x)$, $p = 0$. [yes] $f(x) = x^4+x$, $p = 0$. [no]
nonzero	nonzero	none of the above	

In the ambiguous cases above, we may look at the higher derivatives. For example, if $f'(p) = f''(p) = 0$, then

- If $f^{(3)}(p) \ne 0$, then f has an inflection point at p.
- Otherwise, if $f^{(4)}(p) \ne 0$, then f has a local minimum at p if $f^{(4)}(p) > 0$ and a local maximum if $f^{(4)}(p) < 0$.

An alternative is to look at the first (and possibly second) derivative of f in some small interval around p. This interval may be as small as we wish, as long as its size is greater than 0.

Table 2: Information about f at p from the first and second derivatives in a small interval around p

$f'(p)$	**Change in $f'(x)$ as x moves from left to right of p**	**At p, f has a_____**
0	$f'(x)$ changes from negative to positive at p	Local minimum
0	$f'(x)$ changes from positive to negative at p	Local maximum
0	$f'(x)$ has the same sign on both sides of p. (Implies $f''(p) = 0$.)	Inflection point

$f'(p)$	$f''(p)$	**Change in $f''(x)$ as x moves from left to right of p**	**At p, f has a_____**
nonzero	0	$f''(x)$ changes sign at p.	Inflection point
nonzero	0	$f''(x)$ has the same sign on both sides of p.	None of the above. (However, f' has an inflection point at p.)