Unit 13: Marginal Analysis

1. Business and Economics: $(\text{marginal}) = \begin{pmatrix}\text{instantaneous rate of} \\ \text{change}\end{pmatrix}$

   Mathematics: $(\text{derivative}) = \begin{pmatrix}\text{instantaneous rate} \\ \text{of change}\end{pmatrix}$

2. Thus,

$$(\text{marginal}) = (\text{derivative})$$

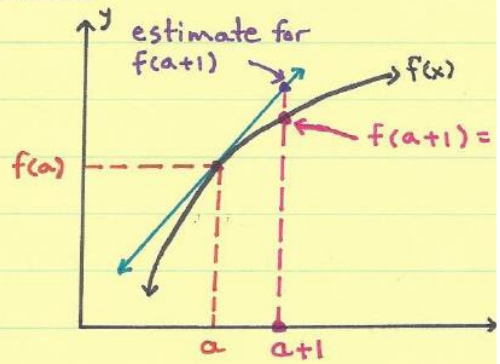
3. <u>Applications:</u>

   a) $C(x)$: cost $\longrightarrow$ $C'(x)$ = marginal cost

   b) $R(x)$: revenue $\longrightarrow$ $R'(x)$ = marginal revenue

   c) $P(x)$: profit $\longrightarrow$ $P'(x)$ = marginal profit

4. A marginal may be used to find an <u>estimate</u> for a function value.



a) <u>Recall:</u>

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

b) Let, $h = 1$. Then,

$$f'(a) \approx \frac{f(a+1) - f(a)}{1}.$$

or,

$$\boxed{f(a+1) \approx f(a) + f'(a)}$$

5. Ex. $C(x)$: the cost of producing x chairs in dollars.

Let,

$$C(x) = 2500 + 5x - 0.04x^2.$$

a) What is the cost of producing 200 chairs?

$a = 200$: $C(200) = 2500 + (5)(200) - 0.04(200)^2$

$$\boxed{C(200) = \$1{,}900}$$

Namely, it costs \$1,900 to produce 200 chairs.

b) Find the marginal cost function.

$$C'(x) = 5 - 0.04(2x)$$

$$\boxed{C'(x) = 5 - 0.08x}$$

c) Find and interpret the marginal cost when the production is 200 chairs.

$$C'(200) = 5 - 0.08(200)$$

$$\boxed{C'(200) = -11} \text{ \$ per chair}$$

(*) Namely, to make one more chair our cost decreases by \$11.

d) Find an estimate for the cost of producing 201 chairs.

$$\begin{aligned} C(201) &\approx C(200) + C'(200) \\ &= 1900 - 11 \\ &= \boxed{\$1{,}889} \end{aligned}$$

e) Find the exact cost of producing 201 chairs.

$$C(201) = 2500 + (5)(201) - 0.04(201)^2 = \boxed{\$1{,}888.96}$$

Definition of marginal

Marginal refers to an instantaneous rate of change, that is, a derivative.

marginal cost = the derivative of the cost function ; $C'(x)$
marginal revenue = the derivative of the revenue function ; $R'(x)$
marginal profit = the derivative of the profit function. ; $P'(x)$

If $f(x)$ is a function of x, the derivative (or marginal f) tells us how $f(x)$ changes if x changes a little.

Marginal cost, revenue, and profit

Marginal Cost

The marginal cost function is just the derivative of the cost function. In business, we use marginal cost $C'(x)$ to approximate the exact cost to produce the $(x+1)$st unit.

find an estimate for

Why does the approximation work? ~~Because~~ Recall:

$$C'(x) = \lim_{h\to 0} \frac{C(x+h)-C(x)}{h}$$

and when $h=1$, the difference quotient equals

$\boxed{C'(x) \approx \frac{C(x+1)-C(x)}{1}}$; $\boxed{C(x+1) \approx C(x) + C'(x)}$

estimate for $C(x+1)$.

X which is the exact cost to produce the $(x+1)$st unit.

approximately

(it should read): $C'(x)$ is approximately the additional cost to produce the $(x+1)$st unit.

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$C(100) = 10000 + 90(100) - 0.05(100)^2$
$= \$18,500$

Example:

$C(101) \approx 18500 + 80 = \boxed{\$18,580}$

Let

$$\boxed{C(x) = 10,000 + 90x - .05x^2}$$

be the total weekly cost (in dollars) of manufacturing x fuel tanks for cars.

$C(101) \approx C(100) + C'(100)$

Compute the marginal cost function and use it to approximate the exact cost to manufacture the 101st fuel tank.

Marginal cost function: $\boxed{C'(x) = 90 - 0.05(2x)} = \boxed{90 - 0.10x}$

Marginal cost at $x = 100$: $C'(100) = 90 - 0.10(100) = 90 - 10 = \boxed{80 \frac{\$}{\text{fuel tank}}}$

X **Summary:** It costs approximately \$80.00 to produce the 101st fuel tank. WRONG!

(it should read) The additional cost to produce the 101-st fuel tank is approximately \$80.

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Example continued $C(x) = 10000 + 90x - 0.05x^2$

The marginal cost function:

$$\boxed{C'(x) = 90 - .10x}$$

Is the cost of a fuel tank increasing, decreasing or remaining the same as x increases?

decreasing b/c we subtract from 90.

The marginal cost function $C'(x)$ gives the (approximate) cost to produce the $(x+1)$st tank.

So the first tank costs about $\boxed{C(0) = 10000}$

See below on page 302.

$\boxed{C'(0) = 90}$ dollars per fuel tank

$C(1) \approx 10{,}000 + 90 = \boxed{10{,}090}$ \$

Second tank cost about

$C'(1) = 90 - 0.10 = \boxed{89.90}$ dollars per fuel tank ; $C(1) = 10000 + 90 - 0.05 = \boxed{\$10{,}089.95}$

Apparently the cost to produce the $(x+1)$st tank is \$.10 less than the cost to produce the xth tank.

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$C(x) = 10000 + 90x - 0.05x^2$

$c'(x) = 90 - 0.10x$

$C(2) \approx 10089.95 + 89.90 = \boxed{\$10{,}179.85}$

Example continued

Compute and interpret $C'(200)$:

$c'(200) = 90 - 0.10(200) = \boxed{\$70 \text{ per fuel tank}}$

i.e., the additional cost to produce the 201-st fuel tank is \$70.

Compute and interpret $C'(500)$:

$c'(500) = 90 - 0.10(500) = \$40$ per fuel tank

i.e., the additional cost to produce the 501-st fuel tank is \$40.

i. $C(0) = 10000$ \$, fixed cost.

ii. $C'(0) = \$90$ per fuel tank.

i.e., the additional cost to make the 1st tank is \$90.

Thus, $C(1) \approx C(0) + C'(0)$

$= 10000 + 90 = \boxed{\$10{,}090}$

i.e., it costs approximately \$10,090 to produce the 1st tank.

iii. $C(1) = 10000 + 90(1) - 0.05(1)^2 = \$10{,}089.95$

$C'(1) = 90 - 0.10(1) = \boxed{\$89.90}$

i.e. the additional cost to produce the 2nd tank is \$89.90.

$C(2) \approx C(1) + C'(1) = 10089.95 + 89.90$

$= \boxed{\$10{,}179.85}$

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Example continued

Reordering to approximate total cost of $x+1$ units from the total cost of x plus marginal cost:

$$C'(x) \doteq C(x+1) - C(x). \quad ✓$$

$$C(x+1) \doteq C(x) + C'(x). \quad ✓$$

Confirm that $C(100) = \$18500$ and recall that $C'(100) = \$80$:

Use this to approximate $C(101)$:

(*)

Summary: [It costs approximately \$80.00 to produce the 101st fuel tank.] Or equivalently, the cost to produce 101 fuel tanks is approximately $C(100) + 80 = \$18580$.

It should read: The additional cost to produce the 101-st fuel tank is approximately \$80.

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Marginal cost for mining coal

The cost (in dollars) to mine x tons of coal is

$$C(x) = 3000 + 100x - 2x^2.$$

Write a formula for the marginal cost function.

$c'(x) = 100 - 4x$

Use the marginal cost to estimate the additional cost to mine 11 tons instead of 10 tons.

$c'(10) = \$60$ per ton

$c'(10) = 100 - 4(10)$
$= 100 - 40$
$= 60$

Marginal cost for mining coal, continued

Problem:

Use the marginal cost to estimate the additional cost to mine 10.2 tons instead of 10 tons.

The additional cost for mining 10.2 tons instead of 10 tons is $C(10.2) - C(10)$. The marginal cost $C'(10)$ approximates the difference quotient

$$\frac{C(10+h) - C(10)}{h}.$$

What is h in this example? $\boxed{h = 0.2}$ $h = 10.2 - 10 = 0.2$

$b - a$

The additional cost to mine 10.2 tons instead of 10 tons of coal is

$$c'(10) \approx \frac{c(10+h) - c(10)}{0.2}$$

$$60 \approx \frac{c(10+h) - c(10)}{0.2}$$

$$(60)(0.2) \approx c(10+h) - c(10)$$

$\boxed{c'(10) \cdot (0.2) = (60)(0.2)}$

$= 12$ \$ per ton

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Marginal Revenue & Profit

The technique of approximating the cost to produce a single item by the marginal cost also applies to revenue and profit.

Revenue: If $R(x)$ is the revenue from selling x units, then the additional revenue earned by selling $x+1$ units rather than x units is:

$\boxed{R(x+1) - R(x) \quad \text{(additional revenue from selling } (x+1)\text{st unit)}}$

The marginal revenue, $R'(x)$ is approximately equal to the additional revenue:

$$\boxed{R'(x) \doteq R(x+1) - R(x).} \; ; \; R(x+1) \approx R(x) + R'(x)$$

Profit: The same thing holds for the profit function, $P(x)$:

$$\boxed{P'(x) \doteq P(x+1) - P(x).} \; ; \; P(x+1) \approx P(x) + P'(x)$$

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Ex. $C(x) = 10000 + 90x - 0.05x^2$

Q. What is the cost to produce 200 fuel tanks?

$C(200) = 10000 + 90(200) - 0.05(200)^2 = \$26{,}000.$

Q. Find the marginal cost function.

$C'(x) = 90 - (0.05)(2x) = 90 - 0.10x$

Q. Find and interpret the marginal at $x = 200$.

$C'(200) = 90 - 0.10(200) = 90 - 20 = 70$ \$ per fuel tank.

The additional cost to produce the 201-st fuel tank is \$70.

Q. Find an <u>estimate</u> for the cost of producing 201 fuel tanks.

$C(201) \approx 26000 + 70 = \boxed{\$26{,}070}$

Application

The price-demand equation and the cost function for the production and sales of television sets are

$$x = 6{,}000 - 30p, \qquad C(x) = 150{,}000 + 20x.$$

where p is the price of a TV and x is the number produced and sold. Profit and cost are in dollars.

a. Express the price p as a function of x: (i.e., isolate p)

$x = 6000 - 30p$; $x - 6000 = -30p$; $\frac{x-6000}{-30} = p$

$\boxed{p = -\frac{1}{30}x + 200}$

b. Express revenue $R(x)$ as a function of x:

$R(x) = x \cdot p(x) = x\left(-\frac{1}{30}x + 200\right) = \boxed{-\frac{1}{30}x^2 + 200x}$

c. Express profit $P(x)$ as a function of x:

$P(x) = R(x) - C(x) = \left(-\frac{1}{30}x^2 + 200x\right) - (150000 + 20x)$

$= -\frac{1}{30}x^2 + 200x - 150000 - 20x = \boxed{-\frac{1}{30}x^2 + 180x - 150000}$

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Application continued

d. Find the marginal cost, marginal revenue, and marginal profit functions:

$C'(x) = 20$

$R'(x) = -\frac{1}{30} \cdot 2x + 200 = -\frac{1}{15}x + 200$

$P'(x) = -\frac{1}{30} \cdot 2x + 180 = -\frac{1}{15}x + 180$

e. Find and interpret $R(3000)$ and $R'(3000)$:

$R(3000) = -\frac{1}{30}(3000)^2 + 200(3000) = -\frac{1}{30}(9000000) + 600000$

$= -300000 + 600000 = \boxed{\$300{,}000}$

The revenue for selling 3000 tv sets is \$300,000.

f. Approximate $R(3001) \doteq R(3000) + R'(3000)$:

$R'(3000) = -\frac{1}{15}(3000) + 200 = -200 + 200 = 0$

The additional revenue from selling the 3001-st tv set is \$0.

$R(3001) \approx 300000 + 0 = \$300{,}000.$

$R(3001) \approx R(3000) + R'(3000)$

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Application continued

g. Find and interpret $P(1500)$ and $P'(1500)$:

$$P(1500) = -\frac{1}{30}(1500)^2 + 180(1500) - 150000 = -\frac{1}{30}(2250000) + 270000 - 150000$$

$$= -75000 + 120000 = \$45,000$$

The profit for selling 1500 tv sets is \$45,000.

h. Approximate $P(1501)$:

$$P'(1500) = -\frac{1}{15}(1500) + 180 = -100 + 180 = \boxed{80 \ \$ \text{ per tv set}}$$

i.e., the additional profit from selling the 1501-st tv set is \$80.

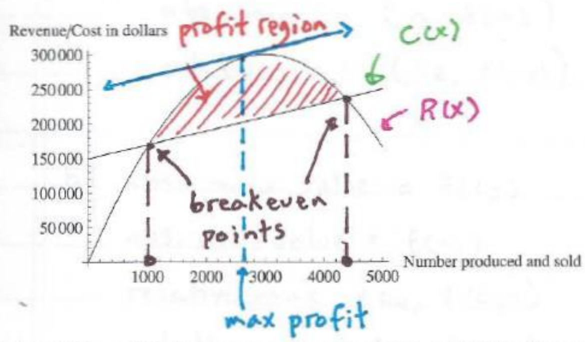
$$P(1501) \approx P(1500) + P'(1500)$$

$$= 45000 + 80$$

$$= \$45,080$$

Application continued

The cost and revenue functions are graphed below.



$P(x) = R(x) - C(x)$

$P'(x) = R'(x) - C'(x)$

At its turning pt (local max), $P'(x) = 0$.

Thus, $R'(x) - C'(x) = 0$.

or, $\boxed{R'(x) = C'(x)}$.

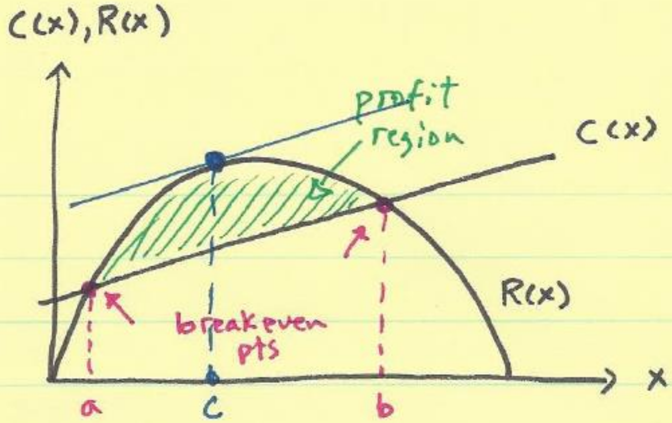
$\boxed{\text{slope of } R(x) = \text{slope of } C(x)}$

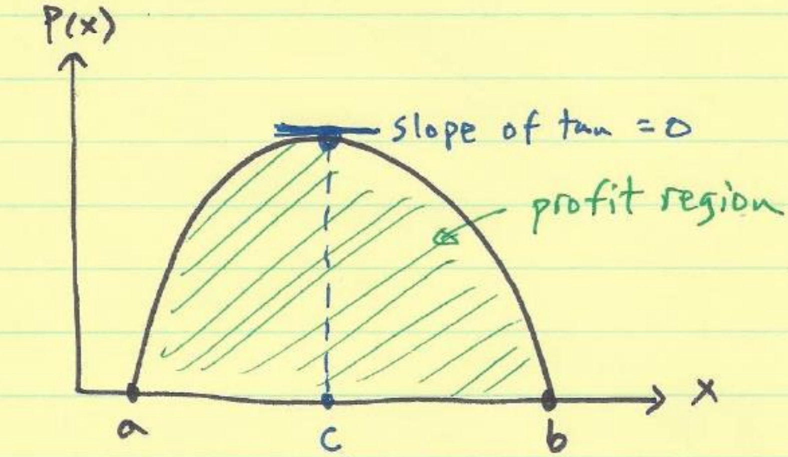
Identify which curve goes with which function.

Shade the profit area.

Mark the break even points.

Mark the points on the revenue and cost graphs where profit is maximized.





$$P(x) = R(x) - C(x)$$

$$P'(x) = R'(x) - C'(x)$$

$$P'(c) = R'(c) - C'(c)$$

$$0 = R'(c) - C'(c)$$

$$\boxed{R'(c) = C'(c)}$$

at the pt where profit is maximized, the slope of $R(x)$ is equal to the slope of $C(x)$.