Handout - Derivative - Chain Rule

Power-Chain Rule a,b are constants.

Function	Derivative	
$y = a \cdot x^n$	$\frac{dy}{dx} = a \cdot n \cdot x^{n-1}$	Power Rule
$y = a \cdot u^n$	$\frac{dy}{dx} = a \cdot n \cdot u^{n-1} \cdot \frac{du}{dx}$	**Power-Chain Rule**

Ex1a. Find the derivative of $y = 8(6x + 21)^8$

Answer: $\boldsymbol{y' = 384(6x + 21)^7}$

$a = 8, \quad n = 8$

$u = 6x + 21 \quad \Rightarrow \frac{du}{dx} = 6$

$\Rightarrow y' = 8 \cdot 8 \cdot (6x + 21)^7 \cdot 6$

Ex1b. Find the derivative of $y = 8\left(4x^2 + 7x + 28\right)^4$

Answer: $\boldsymbol{y' = 32(8x + 7)\left(4x^2 + 7x + 28\right)^3}$

$a = 8, \quad n = 4$

$u = 4x^2 + 7x + 28 \quad \Rightarrow \frac{du}{dx} = 8x + 7$

$\Rightarrow y' = 8 \cdot 4 \cdot \left(4x^2 + 7x + 28\right)^3 \cdot (8x + 7)$

Ex1c. Find the derivative of $y = 2\sqrt{6x^2 + 4x + 26}$

Answer: $\boldsymbol{y' = \dfrac{12x + 4}{\sqrt{6x^2 + 4x + 26}}}$

$a = 2, \quad n = \frac{1}{2}$

$u = 6x^2 + 4x + 26 \quad \Rightarrow \frac{du}{dx} = 12x + 4$

$\Rightarrow y' = 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{6x^2+4x+26}} \cdot (12x + 4)$

Exercises

Find the derivatives of the expressions

a) $5(9x+25)^8$

b) $7(2x+24)^8$

c) $2\left(4x^2+4x+21\right)^9$

d) $6\left(7x^2+4x+22\right)^4$

e) $7\left(7x^2+9x+24\right)^{13/3}$

f) $3\left(7x^2+4x+29\right)^{22/3}$

g) $5\sqrt{x^2+8x+25}$

h) $7\sqrt{7x^2+3x+24}$

i) $\frac{8}{\sqrt{3x^2+3x+22}}$

j) $\frac{3}{\sqrt{2x^2+2x+22}}$

Answers

a) $360(9x+25)^7$

b) $112(2x+24)^7$

c) $18(8x+4)\left(4x^2+4x+21\right)^8$

d) $24(14x+4)\left(7x^2+4x+22\right)^3$

e) $\frac{91}{3}(14x+9)\left(7x^2+9x+24\right)^{10/3}$

f) $22(14x+4)\left(7x^2+4x+29\right)^{19/3}$

g) $\frac{5(2x+8)}{2\sqrt{x^2+8x+25}}$

h) $\frac{7(14x+3)}{2\sqrt{7x^2+3x+24}}$

i) $-\frac{4(6x+3)}{(3x^2+3x+22)^{3/2}}$

j) $-\frac{3(4x+2)}{2(2x^2+2x+22)^{3/2}}$

Sine and Cosine - Chain Rules a, b are constants.

Function	Derivative	
$y = \sin(x)$	$\frac{dy}{dx} = \cos(x)$	Sine Rule
$y = \cos(x)$	$\frac{dy}{dx} = -\sin(x)$	Cosine Rule
$y = a \cdot \sin(u)$	$\frac{dy}{dx} = a \cdot \cos(u) \cdot \frac{du}{dx}$	**Chain-Sine Rule**
$y = a \cdot \cos(u)$	$\frac{dy}{dx} = -a \cdot \sin(u) \cdot \frac{du}{dx}$	**Chain-Cosine Rule**

Ex2a. Find $\frac{dy}{dx}$ where $y = 2\sin\left(9x^3 + 3x^2 + 1\right)$

Answer: $\mathbf{2\left(27x^2 + 6x\right)\cos\left(9x^3 + 3x^2 + 1\right)}$

$a = 2$

$u = 9x^3 + 3x^2 + 1 \quad \Rightarrow \frac{du}{dx} = 27x^2 + 6x$

Ex2b. Find $\frac{dy}{dx}$ where $y = 5\cos\left(9x^5 + 5x^4 + 3\right)$

Answer: $\mathbf{-5\left(45x^4 + 20x^3\right)\sin\left(9x^5 + 5x^4 + 3\right)}$

$a = 5$

$u = 9x^5 + 5x^4 + 3 \quad \Rightarrow \frac{du}{dx} = 45x^4 + 20x^3$

Ex2c. Find the derivative of $y = \left(4\sin(4x) + 4\cos(5x)\right)^4$

Answer: $\mathbf{4 \cdot \left(16\cos(4x) - 20\sin(5x)\right) \cdot \left(4\sin(4x) + 4\cos(5x)\right)^3}$

$n = 4$

$u = 4\sin(4x) + 4\cos(5x)$

$\Rightarrow \frac{du}{dx} = 16\cos(4x) - 20\sin(5x)$

Exercises

Find the derivatives of the expressions

a) $7\sin\left(7x^3+7x^2+8\right)$

b) $5\cos\left(2x^4+7x^3+9\right)$

c) $8\sin\left(8x^3+9x^2+1\right)$

d) $5\cos\left(6x^4+8x^3+2\right)$

e) $\left(\sin(x)+9\cos\left(x^4\right)\right)^4$

f) $\left(2\sin(6x)+9\cos\left(3x^2\right)\right)^3$

g) $\left(4\sin\left(5x^3\right)+8\cos\left(8x^5\right)\right)^5$

h) $\left(\sin\left(7x^3\right)+2\cos\left(6x^4\right)\right)^3$

i) $\left(5\sin\left(5x^5\right)+5\cos\left(7x^4\right)\right)^{\frac{1}{2}}$

j) $\left(6\sin\left(2x^3\right)+5\cos\left(8x^2\right)\right)^{\frac{3}{2}}$

Answers

a) $7\left(21x^2+14x\right)\cos\left(7x^3+7x^2+8\right)$;

b) $-5\left(8x^3+21x^2\right)\sin\left(2x^4+7x^3+9\right)$;

c) $8\left(24x^2+18x\right)\cos\left(8x^3+9x^2+1\right)$;

d) $-5\left(24x^3+24x^2\right)\sin\left(6x^4+8x^3+2\right)$;

e) $4\cdot\left(\cos(x)-36x^3\sin\left(x^4\right)\right)\cdot\left(\sin(x)+9\cos\left(x^4\right)\right)^3$;

f) $3\cdot\left(12\cos(6x)-54x\sin\left(3x^2\right)\right)\cdot\left(2\sin(6x)+9\cos\left(3x^2\right)\right)^2$;

g) $5\cdot\left(60x^2\cos\left(5x^3\right)-320x^4\sin\left(8x^5\right)\right)\cdot\left(4\sin\left(5x^3\right)+8\cos\left(8x^5\right)\right)^4$;

h) $3\cdot\left(21x^2\cos\left(7x^3\right)-48x^3\sin\left(6x^4\right)\right)\cdot\left(\sin\left(7x^3\right)+2\cos\left(6x^4\right)\right)^2$;

i) $\frac{1}{2}\cdot\left(125x^4\cos\left(5x^5\right)-140x^3\sin\left(7x^4\right)\right)\cdot\left(5\sin\left(5x^5\right)+5\cos\left(7x^4\right)\right)^{-\frac{1}{2}}$;

j) $\frac{3}{2}\cdot\left(36x^2\cos\left(2x^3\right)-80x\sin\left(8x^2\right)\right)\cdot\left(6\sin\left(2x^3\right)+5\cos\left(8x^2\right)\right)^{\frac{1}{2}}$;

Exponent and Logarithmic - Chain Rules a,b are constants.

Function	Derivative	
$y = e^x$	$\frac{dy}{dx} = e^x$	Exponential Function Rule
$y = \ln(x)$	$\frac{dy}{dx} = \frac{1}{x}$	Logarithmic Function Rule
$y = a \cdot e^u$	$\frac{dy}{dx} = a \cdot e^u \cdot \frac{du}{dx}$	**Chain-Exponent Rule**
$y = a \cdot \ln(u)$	$\frac{dy}{dx} = \frac{a}{u} \cdot \frac{du}{dx}$	**Chain-Log Rule**

Ex3a. Find the derivative of $y = 6e^{7x+22}$

Answer: $\boldsymbol{y' = 42e^{7x+22}}$
$a = 6$
$u = 7x + 22 \quad \Rightarrow \frac{du}{dx} = 7 \Rightarrow y' = 6 \cdot e^{7x+22} \cdot 7$

Ex3b. Find the derivative of $y = 6e^{7x^2+3x+22}$

Answer: $\boldsymbol{y' = 6 \cdot (14x + 3) \cdot e^{7x^2+3x+22}}$
$a = 6$
$u = 7x^2 + 3x + 22 \quad \Rightarrow \frac{du}{dx} = 14x + 3 \Rightarrow y' = 6 \cdot e^{7x^2+3x+22} \cdot (14x + 3)$

Ex3c. Find the derivative of $y = -2e^{8x^{8/3}+7x^{7/2}}$

Answer: $\boldsymbol{y' = -2 \cdot (\frac{49x^{5/2}}{2} + \frac{64x^{5/3}}{3}) \cdot e^{8x^{8/3}+7x^{7/2}}}$
$a = -2$
$u = 8x^{8/3} + 7x^{7/2} \quad \Rightarrow \frac{du}{dx} = \frac{49x^{5/2}}{2} + \frac{64x^{5/3}}{3}$
$\Rightarrow y' = -2 \cdot e^{8x^{8/3}+7x^{7/2}} \cdot (\frac{49x^{5/2}}{2} + \frac{64x^{5/3}}{3})$

Ex3d. Find the derivative of $y = 7\ln(3x + 26)$

Answer: $\boldsymbol{y' = \dfrac{21}{3x + 26}}$

$a = 7$

$u = 3x + 26 \quad \Rightarrow \frac{du}{dx} = 3 \Rightarrow y' = \frac{7}{3x+26} \cdot 3$

Ex3e. Find the derivative of $y = 8\ln\left(2x^2 + 9x + 26\right)$

Answer: $\boldsymbol{y' = \dfrac{8(4x + 9)}{2x^2 + 9x + 26}}$

$a = 8$

$u = 2x^2 + 9x + 26 \quad \Rightarrow \frac{du}{dx} = 4x + 9$

$\Rightarrow y' = \frac{8}{2x^2+9x+26} \cdot (4x + 9)$

Ex3f. Find the derivative of $y = 7\log\left(9x^{5/2} + 5x^{5/3}\right)$

Answer: $\boldsymbol{y' = 7 \cdot \left(\dfrac{45x^{3/2}}{2} + \dfrac{25x^{2/3}}{3}\right) \cdot \dfrac{1}{9x^{5/2} + 5x^{5/3}}}$

$a = 7$

$u = 9x^{5/2} + 5x^{5/3} \quad \Rightarrow \frac{du}{dx} = \frac{45x^{3/2}}{2} + \frac{25x^{2/3}}{3}$

$\Rightarrow y' = 7 \cdot e^{9x^{5/2}+5x^{5/3}} \cdot \left(\frac{45x^{3/2}}{2} + \frac{25x^{2/3}}{3}\right)$

Exercises

Find the derivatives of the expressions

a) $6e^{8x+26}$

b) $6e^{6x+21}$

c) $4\ln(2x+21)$

d) $6\ln(3x+27)$

e) $6e^{3x^2+8x+27}$

f) $8e^{4x^2+6x+29}$

g) $4\ln\left(4x^2+8x+29\right)$

h) $6\ln\left(7x^2+3x+25\right)$

i) $6e^{5x^{8/3}+5\sqrt{x}}$

j) $9\ln\left(9x^{8/3}+8\sqrt{x}\right)$

Answers

a) $48e^{8x+26}$;

b) $36e^{6x+21}$;

c) $\frac{8}{2x+21}$;

d) $\frac{18}{3x+27}$;

e) $6 \cdot (6x+8) \cdot e^{3x^2+8x+27}$;

f) $8 \cdot (8x+6) \cdot e^{4x^2+6x+29}$;

g) $\frac{4(8x+8)}{4x^2+8x+29}$;

h) $\frac{6(14x+3)}{7x^2+3x+25}$;

i) $6 \cdot \left(\frac{40x^{5/3}}{3} + \frac{5}{2\sqrt{x}}\right) \cdot e^{5x^{8/3}+5\sqrt{x}}$;

j) $9 \cdot \left(24x^{5/3} + \frac{4}{\sqrt{x}}\right) \cdot \frac{1}{9x^{8/3}+8\sqrt{x}}$;

Exercises

Find the derivatives of the expressions

a) $5\sin(5x) + 4e^{4x+1} + \frac{4}{\sqrt{x}}$

b) $2\cos(5x) + 5e^{3x+1} + \frac{8}{\sqrt{x-1}}$

c) $2\cos(4x) + 4\ln(3x-1) - \frac{7}{(3x+1)^{3/5}}$

d) $\cos(5x) + 5e^{x+1} - 7(x-1)^{2/3}$

e) $\left(3\cos(5x) + 2\ln(4x) + 8\sqrt{x}\right)^5$

f) $\left(\cos(4x) + 3\ln(3x) - 7\sqrt[5]{x}\right)^3$

g) $\left(4\sin(3x) + 3\ln(x) - \frac{6}{\sqrt{x}}\right)^{-\frac{1}{2}}$

h) $\left(4\sin(5x) + e^{3x} - \frac{4}{\sqrt[4]{x}}\right)^{\frac{5}{2}}$

Answers

a) $25\cos(5x) + 16e^{4x+1} - \frac{2}{x^{3/2}}$; b) $-10\sin(5x) + 15e^{3x+1} - \frac{4}{(x-1)^{3/2}}$;

c) $-8\sin(4x) + \frac{12}{3x-1} + \frac{63}{5(3x+1)^{8/5}}$; d) $-5\sin(5x) + 5e^{x+1} - \frac{14}{3\sqrt[3]{x-1}}$;

e) $5 \cdot \left(-15\sin(5x) + \frac{2}{x} + \frac{4}{\sqrt{x}} \right) \cdot \left(3\cos(5x) + 2\ln(4x) + 8\sqrt{x} \right)^4$;

f) $3 \cdot \left(-4\sin(4x) + \frac{3}{x} - \frac{7}{5x^{4/5}} \right) \cdot \left(\cos(4x) + 3\ln(3x) - 7\sqrt[5]{x} \right)^2$;

g) $-\frac{1}{2} \cdot \left(12\cos(3x) + \frac{3}{x} + \frac{3}{x^{3/2}} \right) \cdot \left(4\sin(3x) + 3\ln(x) - \frac{6}{\sqrt{x}} \right)^{-\frac{3}{2}}$;

h) $\frac{5}{2} \cdot \left(20\cos(5x) + 3e^{3x} + \frac{1}{x^{5/4}} \right) \cdot \left(4\sin(5x) + e^{3x} - \frac{4}{\sqrt[4]{x}} \right)^{\frac{3}{2}}$;