

Derivatives

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Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{dy}{dx} \right|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x - a)$.

2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.

3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)' = cf'(x)$

2. $(f \pm g)' = f'(x) \pm g'(x)$

3. $(fg)' = f'g + fg'$ - **Product Rule**

4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ - **Quotient Rule**

5. $\frac{d}{dx}(c) = 0$

6. $\frac{d}{dx}(x^n) = nx^{n-1}$ - **Power Rule**

7. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

This is the **Chain Rule**

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Common Derivatives

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad x \neq 0$$

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, \quad x > 0$$

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Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1} f'(x)$
2. $\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$
3. $\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$
5. $\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$
6. $\frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^2[f(x)]$
7. $\frac{d}{dx}(\sec[f(x)]) = f'(x)\sec[f(x)]\tan[f(x)]$
8. $\frac{d}{dx}(\tan^{-1}[f(x)]) = \frac{f'(x)}{1+[f(x)]^2}$

Higher Order Derivatives

The Second Derivative is denoted as

$$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2} \text{ and is defined as}$$

$f''(x) = (f'(x))'$, i.e. the derivative of the first derivative, $f'(x)$.

The n^{th} Derivative is denoted as

$$f^{(n)}(x) = \frac{d^n f}{dx^n} \text{ and is defined as}$$

$f^{(n)}(x) = (f^{(n-1)}(x))'$, i.e. the derivative of the $(n-1)^{\text{st}}$ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $e^{2x-9y} + x^3y^2 = \sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The “trick” is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y' .

$$\begin{aligned} e^{2x-9y} (2 - 9y') + 3x^2y^2 + 2x^3y y' &= \cos(y)y' + 11 \\ 2e^{2x-9y} - 9y'e^{2x-9y} + 3x^2y^2 + 2x^3y y' &= \cos(y)y' + 11 \\ (2x^3y - 9e^{2x-9y} - \cos(y))y' &= 11 - 2e^{2x-9y} - 3x^2y^2 \end{aligned} \quad \Rightarrow \quad y' = \frac{11 - 2e^{2x-9y} - 3x^2y^2}{2x^3y - 9e^{2x-9y} - \cos(y)}$$

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