

Absolute Maximum and Absolute Minimum

Set 1

Example Find the absolute maximum and absolute minimum values of $f(x) = \frac{x}{x^2 + 1}$ on the interval $[0, 2]$

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Answers

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Absolute extrema on a closed interval are found using the Closed Interval Method:

- 1) Find the values of f at the critical numbers of f in (a, b) .
- 2) Find the values of f at the endpoints of the interval.
- 3) The largest of the values from 1) and 2) is the absolute maximum; the smallest of these values is the absolute minimum.

We need the critical numbers. We need to find where $f'(c) = 0$ and where $f'(x)$ does not exist. Since $x^2 + 1 \neq 0$ for real valued x , the derivative always exists.

$$\begin{aligned}f(x) &= \frac{x}{x^2 + 1} \\f'(x) &= \frac{d}{dx} \left[\frac{x}{x^2 + 1} \right] \\&= \frac{(x^2 + 1) \frac{d}{dx} [x] - x \frac{d}{dx} [x^2 + 1]}{(x^2 + 1)^2} \\&= \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} \\&= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \\&= \frac{-x^2 + 1}{(x^2 + 1)^2}\end{aligned}$$

For $f'(c) = 0$, the numerator must equal zero,

$$\begin{aligned}f'(c) = 0 &= \frac{-c^2 + 1}{(c^2 + 1)^2} \\0 &= -c^2 + 1 \\c^2 &= 1 \\c &= \pm 1\end{aligned}$$

We have shown that $f'(1) = 0$ and $f'(-1) = 0$. The critical numbers are $+1, -1$. Only $+1$ is in the interval $(0, 2)$.

Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

$$\begin{aligned}f(+1) &= \frac{1}{(1)^2 + 1} = \frac{1}{2} \\f(0) &= \frac{0}{(0)^2 + 1} = 0 \\f(2) &= \frac{2}{(2)^2 + 1} = \frac{2}{5}\end{aligned}$$

The largest number is $1/2$, so this is the absolute max and it occurs at $x = +1$. The smallest number is 0 , so this is the absolute min and it occurs at $x = 0$.

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Example Find the absolute maximum and absolute minimum values of $f(x) = \frac{\ln x}{x}$ on the interval $[1, 3]$

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- 3) The largest of the values from 1) and 2) is the absolute maximum; the smallest of these values is the absolute minimum.

We need the critical numbers, which means we need to find where $f'(c) = 0$ and where $f'(x)$ does not exist. The only place we could have the derivative not defined is for $x \leq 0$; luckily, this is outside of the interval $(1, 3)$ so we don't need to worry about the derivative being undefined.

$$\begin{aligned}f(x) &= \frac{\ln x}{x} \\f'(x) &= \frac{d}{dx} \left[\frac{\ln x}{x} \right] \\&= \frac{(x) \frac{d}{dx} [\ln x] - \ln x \frac{d}{dx} [x]}{(x)^2} \\&= \frac{(x) \left(\frac{1}{x} \right) - \ln x(1)}{x^2} \\&= \frac{1 - \ln x}{x^2}\end{aligned}$$

For $f'(c) = 0$, the numerator must equal zero,

$$\begin{aligned}f'(c) = 0 &= \frac{1 - \ln c}{c^2} \\0 &= 1 - \ln c \\ \ln c &= 1 \\ c &= e\end{aligned}$$

We have shown that $f'(e) = 0$. The critical number is e , which lies in the interval $(2, 3)$.

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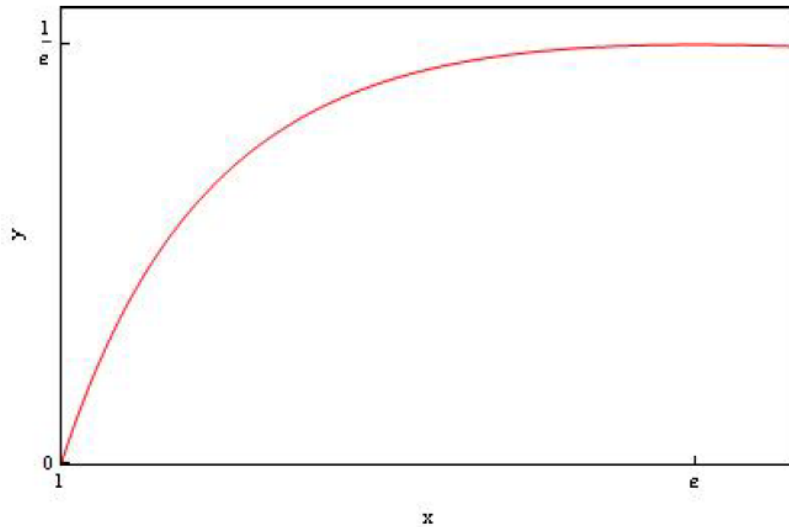
Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

$$\begin{aligned}f(e) &= \frac{\ln e}{e} = \frac{1}{e} \\f(1) &= \frac{\ln 1}{1} = 0 \\f(3) &= \frac{\ln 3}{3}\end{aligned}$$

The smallest number is 0, so this is the absolute min and it occurs at $x = 1$.

It is difficult to determine if $1/e > \ln 3/3$ without resorting to a calculator, or more powerful techniques we have yet to learn. But we can argue based on the properties of derivatives that we *must* have $1/e > \ln 3/3$.

Since the function is continuous in the interval and has a minimum at $x = 1$, and the derivative at $x = e$ is zero means the tangent line is horizontal at $x = e$, and there are no other critical numbers for the function, the function *must* lie below its tangent line near $x = e$. Read that again and see why are the conditions listed are necessary. You might want to draw the function above the tangent line at $x = e$ and see how that leads to contradictions. The function must therefore look something like:



Therefore, the function has an absolute max of $1/e$ at $x = e$.

Example If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$, $0 \leq x \leq 1$.

The derivative will always exist since a and b are positive (if they could be negative, we could have a denominator other than 1).

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The derivative will always exist since a and b are positive (if they could be negative, we could have a denominator other than 1).

The only critical numbers will be if $f'(c) = 0$:

$$\begin{aligned} f(x) &= x^a(1-x)^b \\ f'(x) &= \frac{d}{dx} [x^a(1-x)^b] \\ &= x^a \frac{d}{dx} [(1-x)^b] + (1-x)^b \frac{d}{dx} [x^a] \\ &= bx^a(1-x)^{b-1}(-1) + a(1-x)^b x^{a-1} \\ &= a(1-x)^b x^{a-1} - bx^a(1-x)^{b-1} \\ &= a(1-x)^b x^{a-1} - bx^a(1-x)^{b-1}(1-x)^{-1} \\ &= (1-x)^b x^a (ax^{-1} - b(1-x)^{-1}) \\ &= (1-x)^b x^a \left(\frac{a}{x} - \frac{b}{1-x} \right) \\ &= (1-x)^b x^a \left(\frac{a(1-x) - bx}{x(1-x)} \right) \\ &= (1-x)^{b-1} x^{a-1} (a - (a+b)x) \end{aligned}$$

The critical number is therefore $c = \frac{a}{a+b}$.

Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

$$\begin{aligned} f\left(\frac{a}{a+b}\right) &= \left(\frac{a}{a+b}\right)^a \left(1 - \left(\frac{a}{a+b}\right)\right)^b \\ &= \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b > 0 \text{ (since } a \text{ and } b \text{ are positive)} \\ f(0) &= 0^a(1-0)^b = 0 \\ f(1) &= 1^a(1-1)^b = 0 \end{aligned}$$

Therefore, the maximum value of $f(x) = x^a(1-x)^b$, $0 \leq x \leq 1$ is $\left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b$ which occurs at $x = \frac{a}{a+b}$.