Integration by Substitution

In problems 1 through 8, find the indicated integral.

1. $\int(2x+6)^5dx$
   Solution. Substituting $u = 2x+6$ and $\frac{1}{2}du = dx$, you get

$$\int(2x+6)^5dx = \frac{1}{2}\int u^5du = \frac{1}{12}u^6 + C = \frac{1}{12}(2x+6)^6 + C.$$

2. $\int\left[(x-1)^5 + 3(x-1)^2 + 5\right]dx$
   Solution. Substituting $u = x-1$ and $du = dx$, you get

$$\begin{aligned}\int\left[(x-1)^5 + 3(x-1)^2 + 5\right]dx &= \int(u^5 + 3u^2 + 5)du = \\ &= \frac{1}{6}u^6 + u^3 + 5u + C = \\ &= \frac{1}{6}(x-1)^6 + (x-1)^3 + 5(x-1) + C.\end{aligned}$$

   Since, for a constant C, $C-5$ is again a constant, you can write

$$\int\left[(x-1)^5 + 3(x-1)^2 + 5\right]dx = \frac{1}{6}(x-1)^6 + (x-1)^3 + 5x + C.$$

3. $\int xe^{x^2}dx$
   Solution. Substituting $u = x^2$ and $\frac{1}{2}du = xdx$, you get

$$\int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

4. $\int x^5e^{1-x^6}dx$
   Solution. Substituting $u = 1-x^6$ and $-\frac{1}{6}du = x^5dx$, you get

$$\int x^5e^{1-x^6}dx = -\frac{1}{6}\int e^u du = -\frac{1}{6}e^u + C = -\frac{1}{6}e^{1-x^6} + C.$$

5. $\int\frac{2x^4}{x^5+1}dx$
   Solution. Substituting $u = x^5+1$ and $\frac{2}{5}du = 2x^4dx$, you get

$$\int\frac{2x^4}{x^5+1}dx = \frac{2}{5}\int\frac{1}{u}du = \frac{2}{5}\ln|u| + C = \frac{2}{5}\ln\left|x^5+1\right| + C.$$

6. $\int \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}dx$
Solution. Substituting $u = x^4 - x^2 + 6$ and $\frac{5}{2}du = (10x^3 - 5x)dx$, you get

$$\begin{aligned}\int \frac{10x^3 - 5x}{\sqrt{x^4 - x^2 + 6}}dx &= \frac{5}{2}\int \frac{1}{\sqrt{u}}du = \frac{5}{2}\int u^{-\frac{1}{2}}du = \frac{5}{2} \cdot 2u^{\frac{1}{2}} + C = \\ &= 5\sqrt{x^4 - x^2 + 6} + C.\end{aligned}$$

7. $\int \frac{1}{x \ln x}dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{1}{x \ln x}dx = \int \frac{1}{u}du = \ln |u| + C = \ln |\ln x| + C.$$

8. $\int \frac{\ln x^2}{x}dx$
Solution. Substituting $u = \ln x$ and $du = \frac{1}{x}dx$, you get

$$\int \frac{\ln x^2}{x}dx = \int \frac{2 \ln x}{x}dx = 2\int udu = 2 \cdot \frac{1}{2}u^2 + C = (\ln x)^2 + C.$$

9. Use an appropriate change of variables to find the integral

$$\int (x + 1)(x - 2)^9 dx.$$

Solution. Substituting $u = x - 2$, $u + 3 = x + 1$ and $du = dx$, you get

$$\begin{aligned}\int (x + 1)(x - 2)^9 dx &= \int (u + 3)u^9 du = \int (u^{10} + 3u^9)du = \\ &= \frac{1}{11}u^{11} + \frac{3}{10}u^{10} + C = \\ &= \frac{1}{11}(x - 2)^{11} + \frac{3}{10}(x - 2)^{10} + C.\end{aligned}$$

10. Use an appropriate change of variables to find the integral

$$\int (2x+3)\sqrt{2x-1}dx.$$

Solution. Substituting $u = 2x-1$, $u+4 = 2x+3$ and $\frac{1}{2}du = dx$, you

get

$$\begin{aligned}
\int (2x+3)\sqrt{2x-1}dx &= \frac{1}{2}\int (u+4)\sqrt{u}du = \frac{1}{2}\int u^{\frac{3}{2}}du + 2\int u^{\frac{1}{2}}du = \\
&= \frac{1}{2}\cdot\frac{2}{5}u^{\frac{5}{2}} + 2\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + C = \\
&= \frac{1}{5}(2x-1)^{\frac{5}{2}} + \frac{4}{3}(2x-1)^{\frac{3}{2}} + C = \\
&= \frac{1}{5}(2x-1)^2\sqrt{2x-1} + \frac{4}{3}(2x-1)\sqrt{2x-1} + C = \\
&= (2x-1)\sqrt{2x-1}\left(\frac{2}{5}x - \frac{1}{5} + \frac{4}{3}\right) + C = \\
&= \left(\frac{2}{5}x + \frac{17}{25}\right)(2x-1)\sqrt{2x-1} + C.
\end{aligned}$$