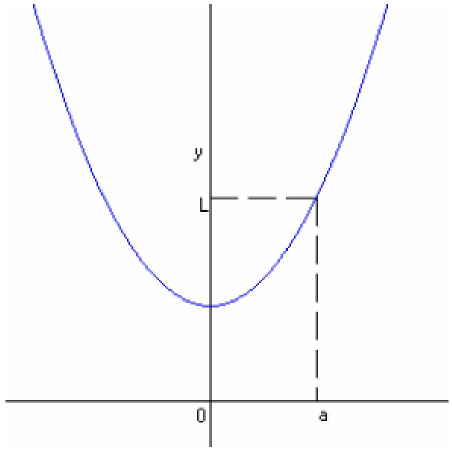
Limits … and How to Find Them

Limit of a function

$\lim\limits_{x \to a} f(x) = L$ if we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.



Example: Let f(x) = $\dfrac{\sqrt{t^2+9}-3}{t^2}$. Discuss the behavior of the values of f(x) when x is close to 0.

Solution: Make a table to see the behavior…

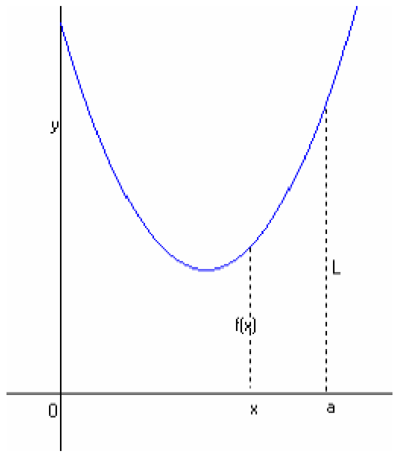
t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.05	0.16666

As t approaches 0, the values of the function seem to approach 0.16666…

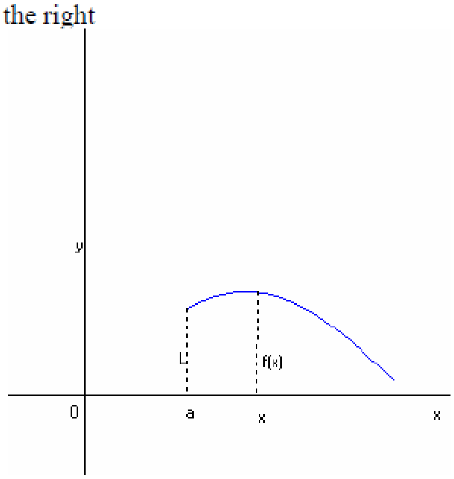
$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = 0.16666$$

One-Sided Limits

Definition: We write $\lim_{x \to a^-} f(x) = L$ and say the **left-hand limit** of f(x) as x approaches a is equal to L if we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to a and **less** than a i.e x approaches a from the left.



Definition: We write $\lim_{x \to a^+} f(x) = L$ and say the **right-hand limit** of f(x) as x approaches a is equal to L if we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to a and **greater** than a i.e x approaches a from the right

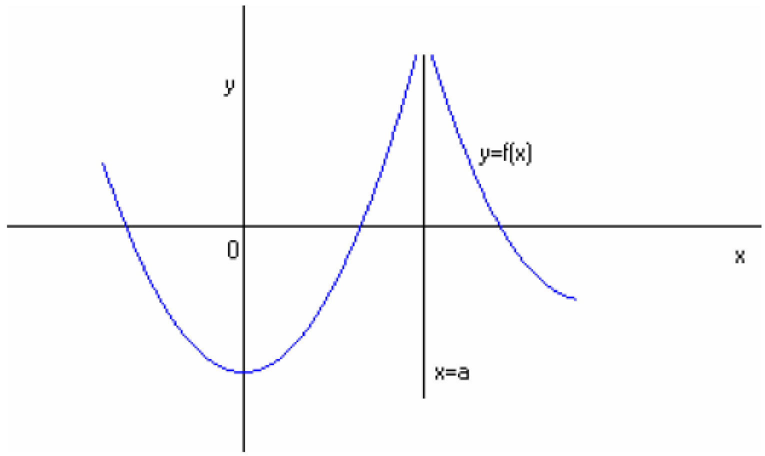


$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Infinite limits

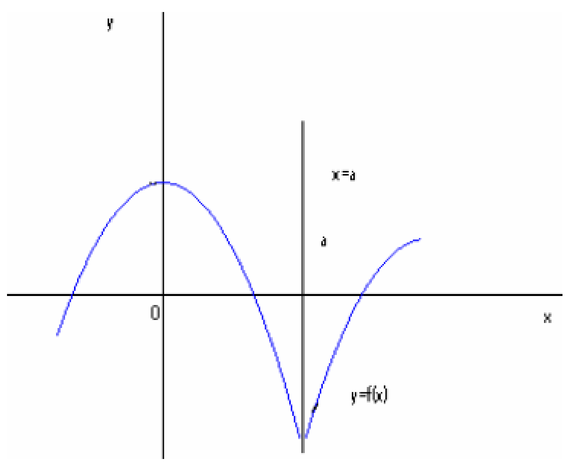
Definition: Let f be a function defined on both sides of a. Then

$\lim_{x \to a} f(x) = \infty$ means that the values of f(x) can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.



Definition: Let f be a function defined on both sides of a. Then

$\lim_{x \to a} f(x) = -\infty$ means that the values of f(x) can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

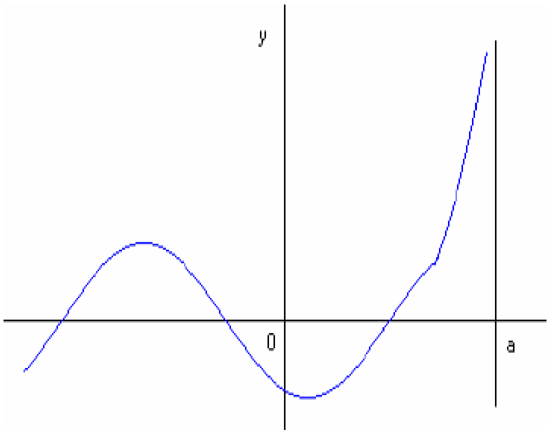


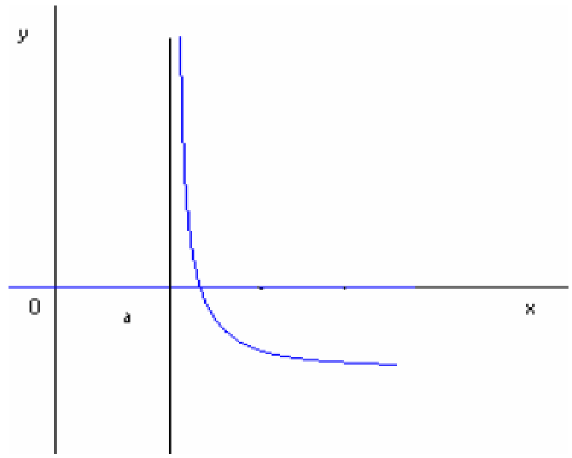
Vertical Asymptotes

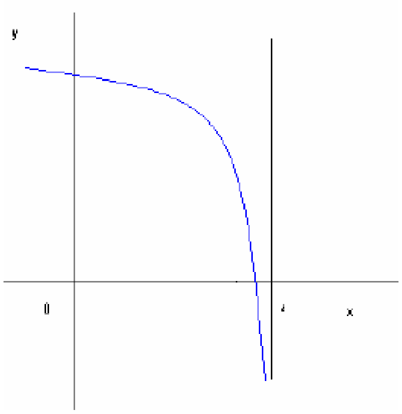
Definition: The line x = a is called a vertical asymptote of the curve y = f(x) if at least one of the following statements is true:

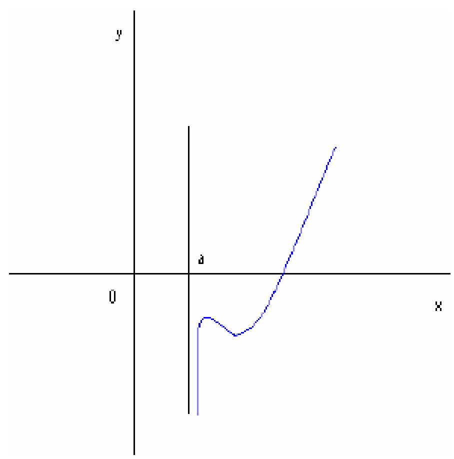
$\lim_{x \to a} f(x) = \infty$ $\lim_{x \to a^-} f(x) = \infty$ $\lim_{x \to a^+} f(x) = \infty$

$\lim_{x \to a} f(x) = -\infty$ $\lim_{x \to a^-} f(x) = -\infty$ $\lim_{x \to a^+} f(x) = -\infty$







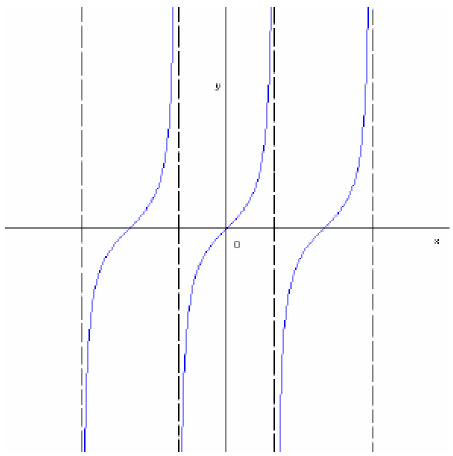


Example: Find the vertical asymptotes of f(x) = tanx

Solution: Because $\tan x = \dfrac{\sin x}{\cos x}$

There are potential vertical asymptotes where cosx = 0. In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas sin x is positive when x is near $\pi/2$, we have $\lim_{x\to(\pi/2)^-} \tan x = \infty$ and $\lim_{x\to(\pi/2)+} \tan x = -\infty$. This shows that the line $x = \pi/2$ is a vertical asymptote.

Similar reasoning shows that the lines $x = (2n+1)\pi/2$, (odd multiples of $\pi/2$),where n is an integer, are all vertical asymptotes of f(x) = tanx.



Calculating Limits Using the Limit Laws

Suppose that c is a constant and the limits
$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$
5. $\lim_{x \to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \text{ if } \lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$ where n is a positive integer.
7. $\lim_{x \to a} c = c$ 8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$, where n is a positive integer.
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer.
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, where n is a positive integer.

Evaluate $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

Solution:
$$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$$
$$= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x}$$
$$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}$$
$$= -\frac{1}{11}$$

Example. Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$

Solution: Let f(x) = $\frac{x^2-1}{x-1}$. We can't find the limit by substituting x=1 because f(1) isn't defined. Nor can we apply the quotient rule because the limit of the denominator is 0. Instead we factor the numerator as a difference of squares:

$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$. The numerator and denominator have a common factor of x-1. When we take the limit as x approaches 1, we have $x \neq 1$ so $x-1 \neq 0$. Therefore, we can cancel the common factor and compute the limit as follows

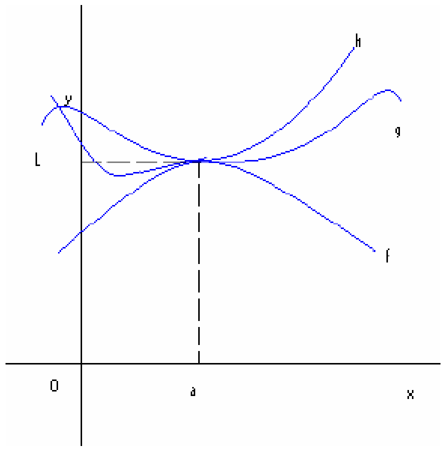
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$
$$= \lim_{x\to 1}(x+1)$$
$$= 1 + 1 = 2$$

The Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ when x is near a and

$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then

$\lim_{x \to a} g(x) = L$.



Example. Show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$.

Solution: First note that we cannot use $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0 = \lim_{x \to 0} x^2 . \lim_{x \to 0} \sin\frac{1}{x}$ because $\lim_{x \to 0} \sin\frac{1}{x}$ does not exist. However since $-1 \le \sin\frac{1}{x} \le 1$,

Multiply all sides by x^2

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2 .$$

We know that $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} (-x^2) = 0$

Taking $f(x) = -x^2, g(x) = x^2 \sin(\frac{1}{x}), h(x) = x^2$ in the squeeze theorem, we obtain

$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

The Precise Definition of a Limit

Let f be a function on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of f(x) as x approaches a is L, and we write

$\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Example prove that $\lim_{x \to 3}(4x-5)=7$.

Solution

1. Preliminary analysis of the problem (guessing a value for δ). Let ε be a given positive number. We want to find a number δ such that

   $|(4x-5)-7|<\varepsilon$ whenever $0<|x-3|<\delta$

But $|(4x-5)-7|=|4x-12|=|4(x-3)|=4|x-3|$. Therefore, we want $4|x-3|<\varepsilon$ whenever $0<|x-3|<\delta$ that is, $|x-3|<\dfrac{\varepsilon}{4}$ whenever $0<|x-3|<\delta$. This suggests that we should choose $\delta = \varepsilon/4$.

2. Proof (showing that this δ works). Given $\varepsilon>0$, choose $\delta=\varepsilon/4$. If $0<|x-3|<\delta$, then

   $$|(4x-5)-7|=|4x-12|=|4(x-3)|=4|x-3|<4\delta=4(\frac{\varepsilon}{4})=\varepsilon$$

   Thus $|(4x-5)-7|<\varepsilon$ whenever $0<|x-3|<\delta$

   Therefore by definition of a limit

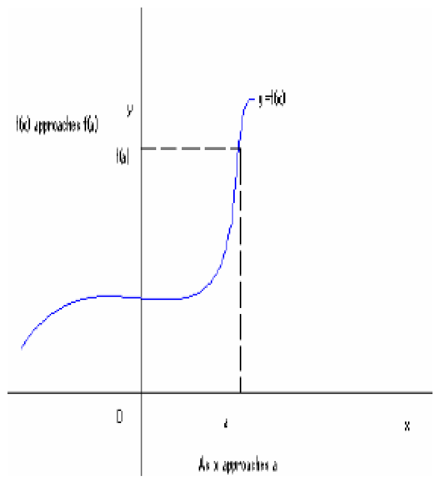
   $$\lim_{x \to 3}(4x-5)=7 .$$

Continuity

A function f is continuous at a number a if
$$\lim_{x \to a} f(x) = f(a)$$

Notice that the definition implicitly requires three things if f is continuous at a:

1. f(a) is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$



A function f is continuous on an interval if it is continuous at every number in the interval.

Example: Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1,1].
Solution: If $-1 < a < 1$, then using the Limit Laws, we have

$$\underset{x \to a}{Lim}\, f(x) = \lim_{x \to a}(1 - \sqrt{1 - x^2})$$

$$= 1 - \lim_{x \to a}\sqrt{1 - x^2}$$

$$= 1 - \sqrt{\lim_{x \to a}(1 - x^2)}$$

$$= 1 - \sqrt{1 - a^2}$$

$$= f(a)$$

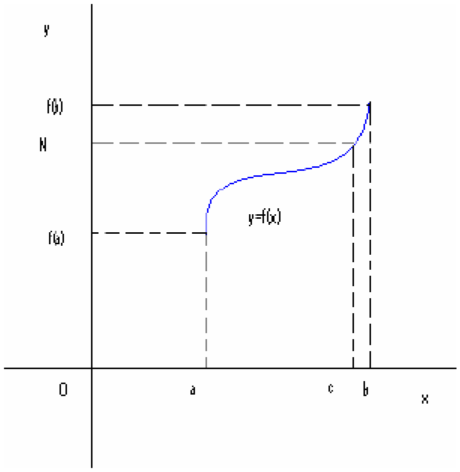
Therefore f is continuous on [-1.1].

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:
1. f + g   2. f – g   3. cf   4 fg   5. f/g if $g(a) \neq 0$

Any polynomial is continuous everywhere, that is it continuous on $\Re = (-\infty, \infty)$
Any rational function is continuous wherever it is defined, that is, it is continuous on its domain.

Intermediate Value Theorem

Suppose that f is continuous on the closed interval [a,b] and let N be any number between f(a) and f(b), where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that f(c) = N.



Example. Using the Intermediate Value Theorem, let's
Show that there is a root of the equation
$4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Solution: let f(x) = $4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that f(c)= 0. Therefore, we take a=1, b=2, N=0.
We have f(1) = 4 – 6 + 3 – 2 = -1 < 0
f(2) = 32 – 24 + 6 – 2 = 12 > 0
Thus, f(1) < 0 < f(2); that is, N = 0 is a number between f(1) and f(2). Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that f(c) = 0. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval (1,2).

References:

Calculus, James Stewart 5th edition