Limits … and How to Find Them

8. Calculus: Limits and Derivatives

This section contains review material on:

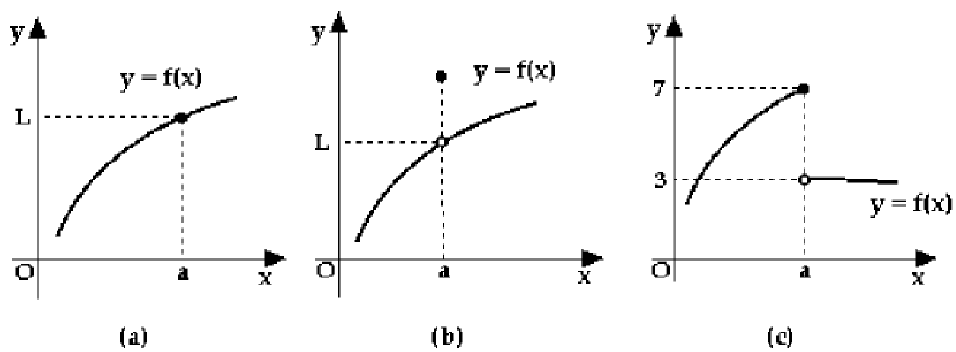
- Limits
- Derivatives

Limits. We do not intend to go into theoretical considerations about limits and other concepts of calculus, but rather concentrate on a few basic (mostly technical) issues.

We say that the limit of $f(x)$, as x approaches a, is L, and write $\lim_{x\to a} f(x) = L$, if we can make the values $f(x)$ as close to L as needed by choosing the values for x inside a small enough interval around a (for various reasons we require that $x \neq a$).

This statement is far from a precise definition, but is a good one to start with; it enables us to develop intuitive understanding of limits for functions of one variable.

Consider the following graphs.



For functions in (a) and (b), $\lim_{x\to a} f(x) = L$. According to our definition, the behaviour of f at a is irrelevant for its limit as x approaches a (remember, in the definition we required that $x \neq a$). Thus, the function in (b) would have had a limit equal to L even if it were not defined at a.

Consider the case (c). Can the limit of $f(x)$ as x approaches a be 7?

The answer is no — for the following reason: no matter how small interval around a we take, there will always be values of x (in this case, to the right of a, inside the interval) for which the function is approximately equal to 3 — and that is not close to 7.

Using a similar argument, we could rule out any other real number as a value of the limit of $f(x)$ as x approaches a. In such cases, we say that the limit does not exist.

Limits … and How to Find Them

Algebraically, we compute limits using limit laws

Limit laws

Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist; then

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad \text{if } \lim_{x\to a} g(x) \neq 0$$

There are many more laws, most of which boil down to the following. Recall that an algebraic function is a function that is built from polynomials by using elementary algebraic operations and by taking roots. Then

If $f(x)$ is an algebraic function and $f(a)$ is defined, then $\lim_{x\to a} f(x) = f(a)$

So, in some cases it is possible to compute limits by substituting a for x.

Example 1. Compute $\lim_{x\to 3} \frac{\sqrt{x}+3x}{x^2-4x+4}$.

Solution

Given function is an algebraic function; thus,

$$\lim_{x\to 3} \frac{\sqrt{x}+3x}{x^2-4x+4} = \frac{\sqrt{3}+3(3)}{(3)^2-4(3)+4} = \frac{\sqrt{3}+9}{1} = \sqrt{3}+9. \quad \blacksquare$$

Exercise 1. Compute the following limits.

(a) $\lim_{x\to -2} \frac{x^2-4x+2}{x-2}$ (b) $\lim_{x\to 0} \frac{\sqrt[3]{x^2+x+1}-1}{\sqrt{x}+1}$. ♣

In some cases, we have to simplify an expression before taking limits. Let us consider a few examples.

Example 2. Compute the following limits.

(a) $\lim_{x\to 1} \frac{x^2-1}{x-1}$ (b) $\lim_{x\to 0} \frac{x^3-x}{x}$ (c) $\lim_{x\to 3} \frac{x^2+x-12}{x^2-9}$.

Solution

(a) Substituting $x=1$, we get $\frac{x^2-1}{x-1} = \frac{0}{0}$, which is not defined (such expression is called an indeterminate form). Notice that it is possible to cancel the fraction:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2.$$

(b) As in (a), cancel the fraction:

$$\lim_{x\to 0} \frac{x^3-x}{x} = \lim_{x\to 0}(x^2-1) = -1.$$

(c) Factoring both the numerator and the denominator, we get

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x+4}{x+3} = \frac{7}{6}. \quad \blacksquare$$

Exercise 2. Compute the following limits.

(a) $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1}$ (b) $\lim_{x\to 0} \dfrac{x^2+4x-21}{x^2-49}$ (c) $\lim_{x\to -7} \dfrac{x^2+4x-21}{x^2-49}$.

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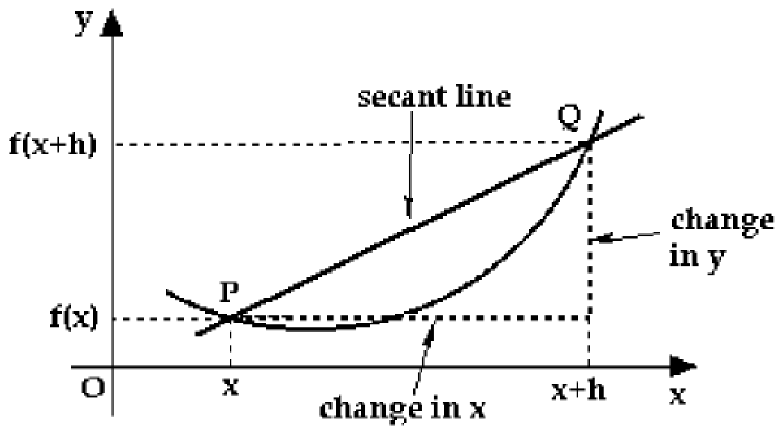
Tangent and Derivative. Consider the graph of a function $y = f(x)$, and pick a point $P(x, f(x))$ on it.



Choose a nearby value of the variable, call it $x + h$ ($x + h$ is h units away from x; "nearby" means that h is small). The corresponding value of the function is $f(x + h)$. Now, we have two points on the curve: $P(x, f(x))$ and $Q(x + h, f(x + h))$. The slope of the line joining these two points (this line is called a secant line) is given by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}.$$

Now imagine that h gets closer and closer to zero, so that $x + h$ approaches x. In other words, imagine that the point Q slides along the curve towards the point P. The limiting position of the secant lines (joining P and Q) as Q approaches P is called the tangent line to the curve $y = f(x)$ at $(x, f(x))$. Its slope is given by

$$m = \text{slope of the tangent} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit in question exists.

This number is also called the derivative of $f(x)$ at x, and is denoted by $f'(x)$. Thus,

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

By computing $f'(x)$ at all x where that is possible, we obtain the derivative function. Thus, the derivative of a function is another function. The value of the derivative at a particular point is equal to the slope of the tangent line at that point.

Example 3. Find the equation of the line tangent to the graph of $y = x^2$ at the point $(1, 1)$.

Solution

To get a line, we need a point (we have it) and a slope. The slope of a tangent is given by $m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, where $f(x) = x^2$ and $x = 1$. Thus,

$$m = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0}(2 + h) = 2.$$

It follows that the equation of the desired tangent line is $y - 1 = 2(x - 1)$; i.e., $y = 2x - 1$. ∎

Example 4. Find the equation of the line tangent to the graph of $y = 1/x$ at the point where $x = 2$.

Solution

The point of tangency has coordinates $x = 2$ and $y = 1/x = 1/2$. To get the slope, we substitute $f(x) = 1/x$ and $x = 2$ into the definition:

$$m = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} = \lim_{h\to 0} \frac{-h}{2(2+h)} \frac{1}{h} = \lim_{h\to 0} \frac{-1}{2(2+h)} = \frac{-1}{4}.$$

Thus, the equation of the tangent is $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + 1$. ∎

Exercise 3. Find the equation of tangent to the graph of $y = 1/x^2$ at the point where $x = 1$.

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Example 5. Using the definition, compute the derivative of $f(x) = \sqrt{x}$.

Solution

The derivative of $f(x) = \sqrt{x}$ is given by

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

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Exercise 4. Using the definition, compute the derivatives of

(a) $y = \sqrt{x + 1}$ (b) $y = 1/\sqrt{x}$

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Let $y = f(x)$. Besides $f'(x)$, commonly used notation for derivatives includes y', $\frac{dy}{dx}$ and $\frac{df}{dx}$. Using the definition of the derivative, we could derive the following differentiation formulas (c and n denote constants).

Derivative of constant functions and of powers of x

$$\text{if } f(x) = c, \text{ then } f'(x) = 0; \qquad \text{in short, } c' = 0$$

$$\text{if } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}; \qquad \text{in short, } (x^n)' = nx^{n-1}$$

Let $f(x)$ and $g(x)$ be two functions and denote by $f'(x)$ and $g'(x)$ their derivatives.

$$(f(x) \pm g(x))' = f'(x) \pm g'(x) \qquad \text{(sum and difference rules)}$$

$$(cf(x))' = cf'(x) \qquad \text{(constant times function rule)}$$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \qquad \text{(product rule)}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \qquad \text{(quotient rule)}$$

Example 6. Compute the derivatives of the following functions.

(a) $f(x) = 6x^2 + 7x + 4$ (b) $f(x) = x^3 + \dfrac{1}{x^3}$ (c) $f(x) = \sqrt{5}\,x + \sqrt{5x}$

(d) $y = x^{\sqrt{5}}$ (e) $y = \dfrac{\sqrt{3}}{x^{10}}$ (f) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$.

Solution

(a) Using the sum rule and the constant times function rule, we get

$$f'(x) = 6 \cdot 2x + 7 \cdot 1 + 0 = 12x + 7.$$

(b) Write $f(x) = x^3 + x^{-3}$; thus, $f'(x) = 3x^2 - 3x^{-4}$.

(c) Rewrite $f(x)$ as $f(x) = \sqrt{5}\,x + \sqrt{5}\sqrt{x} = \sqrt{5}\,x + \sqrt{5}x^{1/2}$. Thus,

$$f'(x) = \sqrt{5} \cdot 1 + \sqrt{5}\tfrac{1}{2}\,x^{-1/2} = \sqrt{5} + \frac{\sqrt{5}}{2\sqrt{x}}.$$

(d) Since $\sqrt{5}$ is a constant, we apply the x^n rule with $n = \sqrt{5}$; thus, $y' = \sqrt{5}x^{\sqrt{5}-1}$.

(e) Write $y = \sqrt{3}x^{-10}$; it follows that $y' = \sqrt{3}(-10)x^{-11}$.

(f) $f(x) = x^{3/2} + x^{2/3}$; thus, $f'(x) = \frac{3}{2}x^{1/2} + \frac{2}{3}x^{-1/3}$. ∎

Exercise 5. Compute the derivatives of the following functions.

(a) $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$ (b) $f(x) = \dfrac{6}{\sqrt[3]{x^4}}$ (c) $y = \dfrac{x^2 + 1}{\sqrt{x}}$

(d) $y = x^2 + \pi^2 + x^{\pi}$.

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Example 7. Find the equation of the line tangent to the curve $y = \dfrac{1}{x^4 + x^2 + x + 1}$ at the point where $x = 0$.

Solution

Substituting $x = 0$ into the formula for y, we get $y = 1$; so, the point of tangency is $(0, 1)$. The slope of the tangent line is given by $m = y'(0)$. Using the quotient rule,

$$y' = \frac{0(x^4 + x^2 + x + 1) - 1(4x^3 + 2x + 1)}{(x^4 + x^2 + x + 1)^2} = -\frac{4x^3 + 2x + 1}{(x^4 + x^2 + x + 1)^2}.$$

It follows that $y'(0) = -1$; the equation of the tangent is $y - 1 = -1(x - 0)$, i.e., $y = -x + 1$. ∎

Exercise 6. Find the equation of the line tangent to the curve $y = \dfrac{x + 3}{x^2 + x + 3}$ at the point where $x = 0$.

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The derivatives of exponential, logarithmic and trigonometric functions are given below.

$$(e^x)' = e^x \qquad (a^x)' = a^x \ln a \qquad (\ln x)' = \frac{1}{x} \qquad (\log_a x)' = \frac{1}{x \ln a}$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\tan x)' = \sec^2 x$$

$$(\csc x) = -\csc x \cot x \qquad (\sec x)' = \sec x \tan x \qquad (\cot x)' = -\csc^2 x$$

Example 8.

(a) Compute y' if $y = x^2 \sec x$.

(b) Compute y' if $y = \dfrac{\cos x - 1}{\sin x}$.

(c) Derive the formula for $(\tan x)'$.

Solution

(a) Using the product rule, we get $y' = 2x \sec x + x^2 \sec x \tan x$.

(b) By the quotient rule,

$$y' = \frac{-\sin x \ \sin x - (\cos x - 1)\cos x}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x + \cos x}{\sin^2 x} = \frac{\cos x - 1}{\sin^2 x}.$$

(c) Applying the quotient rule,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \ \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{1}{\cos^2 x} = \sec^2 x.$$

∎

Exercise 7.

(a) Compute y' if $y = \dfrac{\tan x}{\sec x}$.

(b) Compute y' if $y = 3 \sin x \tan x + 3$.

(c) Derive the formula $(\sec x)' = \sec x \tan x$.

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Chain Rule. The derivative of the composition of two functions is computed as a product of their derivatives. To be precise: Let $f(x)$ and $g(x)$ be two functions and let $(g \circ f)(x) = g(f(x))$ be their composition.

Chain rule (version I)
If $h(x) = g(f(x))$, then $h'(x) = g'(f(x))\, f'(x)$

Note: in computing the composition $g(f(x))$, we apply f to x first, and then we apply g to $f(x)$. According to the chain rule, when doing the derivative, we proceed in the opposite order: g is done first, and then f. One more thing: the $f(x)$ part in $g'(f(x))$ states that, while doing the derivative of g, we do not change $f(x)$ (the $f(x)$ term is usually called the "inside").

Sometimes we think of the composition in the following way: $y = g(u)$ and $u = f(x)$ (i.e., y depends on u, and u depends on x). In that case, y depends on x and its derivative is

Chain rule (version II)
If $y = g(u)$ and $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

Example 9. Compute the derivatives of the following functions.

(a) $y = (x^2+1)^{14}$ (b) $y = \sqrt{\sin x + 1}$ (c) $y = \dfrac{1}{e^x+2}$

(d) $y = \sin(x^2+1)$ (e) $y = \cos(\sec x)$ (f) $y = (\sin x)^2 + \sin(x^2)$.

Solution

(a) We start by computing the derivative of the power of 14 :

$$y' = 14(x^2+1)^{13}(x^2+1)' = 14(x^2+1)^{13}\,2x = 28x(x^2+1)^{13}.$$

(b) Writing $y = (\sin x + 1)^{1/2}$, we get

$$y' = \tfrac{1}{2}(\sin x+1)^{-1/2}(\sin x+1)' = \tfrac{1}{2}(\sin x+1)^{-1/2}\ \cos x = \tfrac{1}{2}\cos x(\sin x+1)^{-1/2}.$$

(c) $y = (e^x+2)^{-1}$; thus,

$$y' = (-1)(e^x+2)^{-2}(e^x+2)' = -e^x(e^x+2)^{-2} = -\frac{e^x}{(e^x+2)^2}.$$

(d) We start by computing the derivative of sin:

$$y' = \cos(x^2+1)\ (x^2+1)' = 2x\cos(x^2+1).$$

(e) $y' = -\sin(\sec x)\ (\sec x)' = -\sin(\sec x)\sec x\tan x$.

(f) We have to be careful about the order:

$$y' = 2(\sin x)^1(\sin x)' + \cos(x^2)\ (x^2)' = 2\sin x\cos x + 2x\cos(x^2) = \sin 2x + 2x\cos(x^2).$$

In simplifying, we used the formula $2\sin x\cos x = \sin 2x$. ∎

Exercise 8. Compute the derivatives of the following functions.
(a) $y = \dfrac{1}{x^3 + x - 2}$ (b) $y = (\sqrt{x} + 1)^2$ (c) $\tan(x^2) + \tan(x^2 + 1)$
(d) $y = \sec(e^x)$ (e) $y = \cos^2(x^2)$ (f) $y = x^2 \sin(1/x)$.

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Example 10. Compute the derivatives of the following functions.
(a) $y = e^{4x+2}$ (b) $y = \ln(\sin x + 2)$ (c) $y = 2^{3x}$
(d) $y = \ln(x^2 + 3x + e^x)$ (e) $y = e^{\sin x} + \sin(e^x)$ (f) $y = \sec\sqrt{x^2 + x}$.

Solution

(a) We start by doing the derivative of the exponential function:

$$y' = e^{4x+2}(4x + 2)' = 4e^{4x+2}.$$

(b) The derivative of $\ln x$ is $1/x$; thus,

$$y' = \frac{1}{\sin x + 2}(\sin x + 2)' = \frac{\cos x}{\sin x + 2}.$$

(c) The derivative of 2^x is $2^x \ln 2$; it follows that

$$y' = 2^{3x} \ln 2\ (3x)' = 3 \cdot 2^{3x} \ln 2.$$

(d) As in (b),

$$y' = \frac{1}{x^2 + 3x + e^x}(x^2 + 3x + e^x)' = \frac{2x + 3 + e^x}{x^2 + 3x + e^x}.$$

(e) $y' = e^{\sin x}(\sin x)' + \cos(e^x)(e^x)' = \cos x\, e^{\sin x} + e^x \cos(e^x)$.

(f) Write $y = \sec(x^2 + x)^{1/2}$ and recall that $(\sec x)' = \sec x \tan x$. Then

$$\begin{aligned} y' &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2}((x^2 + x)^{1/2})' \\ &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2} \tfrac{1}{2}(x^2 + x)^{-1/2}(2x + 1). \end{aligned}$$

∎

Exercise 9. Compute the derivatives of the following functions.
(a) $y = \ln\sqrt{x} + \sqrt{\ln x}$ (b) $y = \dfrac{1 - \ln x}{1 + \ln x}$ (c) $f(x) = 2^{e^x}$
(d) $y = e^x + x^e + e^e$ (e) $y = \sec^2(\ln x)$

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Exercise 8. Compute the derivatives of the following functions.

(a) $y = \dfrac{1}{x^3 + x - 2}$ (b) $y = (\sqrt{x} + 1)^2$ (c) $\tan(x^2) + \tan(x^2 + 1)$

(d) $y = \sec(e^x)$ (e) $y = \cos^2(x^2)$ (f) $y = x^2 \sin(1/x)$.

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Solution

(a) We start by doing the derivative of the exponential function:

$$y' = e^{4x+2}(4x + 2)' = 4e^{4x+2}.$$

(b) The derivative of $\ln x$ is $1/x$; thus,

$$y' = \frac{1}{\sin x + 2}(\sin x + 2)' = \frac{\cos x}{\sin x + 2}.$$

(c) The derivative of 2^x is $2^x \ln 2$; it follows that

$$y' = 2^{3x} \ln 2\ (3x)' = 3 \cdot 2^{3x} \ln 2.$$

(d) As in (b),

$$y' = \frac{1}{x^2 + 3x + e^x}(x^2 + 3x + e^x)' = \frac{2x + 3 + e^x}{x^2 + 3x + e^x}.$$

(e) $y' = e^{\sin x}(\sin x)' + \cos(e^x)(e^x)' = \cos x e^{\sin x} + e^x \cos(e^x)$.

(f) Write $y = \sec(x^2 + x)^{1/2}$ and recall that $(\sec x)' = \sec x \tan x$. Then

$$\begin{aligned} y' &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2}((x^2 + x)^{1/2})' \\ &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2} \tfrac{1}{2}(x^2 + x)^{-1/2}(2x + 1). \end{aligned}$$

∎

Exercise 9. Compute the derivatives of the following functions.

(a) $y = \ln\sqrt{x} + \sqrt{\ln x}$ (b) $y = \dfrac{1 - \ln x}{1 + \ln x}$ (c) $f(x) = 2^{e^x}$

(d) $y = e^x + x^e + e^e$ (e) $y = \sec^2(\ln x)$

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Example 11. Find dy/dx for the following functions.

(a) $y = 4u^2 - 3u + 2,\ u = e^x + 2e^{2x}$ (b) $y = \ln u,\ u = \sin x + \cos x$

Solution

(a) By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = (8u - 3)(e^x + 4e^{2x}) = 8((e^x + 2e^{2x}) - 3)(e^x + 4e^{2x}).$$

(b) As in (a),

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{u}(\cos x - \sin x) = \frac{\cos x - \sin x}{\sin x + \cos x}.$$

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Exercise 10. Compute the derivatives of the following functions.

(a) $y = \sqrt{u^2 + 2},\ u = \cot x$ (b) $y = \log_2 u,\ u = e^x + 4$

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