Formal Definition: Limits

Limits are more formally defined as

"L is the limit of f(x) as x approaches a if for every number $\in > 0$, there is a corresponding number $\delta > 0$ such that for all x. Using notation we write

$$\lim_{x \to a} f(x) = L \quad \text{IFF} \quad 0 < |x - a| < \delta \ \rightarrow \ |f(x) - L| < \in$$

From the formal definition, right-handed limit can be defined as:

$$0 < x - a < \delta \ \rightarrow \ |f(x) - L| < \in$$

And written as:

$$\lim_{x \to a^+} f(x) = L$$

Whereas the left-handed limit can be defined as:

$$-\delta < x - a < 0 \ \rightarrow \ |f(x) - L| < \in$$

And written as:

$$\lim_{x \to a^-} f(x) = L$$

Example 1: Testing the Definition

Show: $\lim_{x\to 1}(5x-3)=2$

We have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ, that is if:

$$0 < |x-1| < \delta$$

Then $f(x)$ is within distance $\in$ of $L=2$, that is

$$|f(x)-2| < \in$$
$$|(5x-3)-2| < \in$$
$$|5x-5| < \in$$
$$5|x-1| < \in$$
$$|x-1| < \frac{\in}{5}$$

Thus, we can take $\delta = \frac{\in}{5}$ due to the fact that $0 < |x-1| < \delta$, then:

$$|f(x)-2| < \in$$

So, if $|(5x-3)-2| = |5x-5| = 5|x-1| < 5(\delta) = 5\left(\frac{\in}{5}\right) = \in$

therefore, $0 < |x-1| < \frac{\in}{5} \rightarrow |f(x)-2| < \in$ and $\lim_{x\to 1}(5x-3)=2$.