Limits: Advanced Topics

Previously, when we found that the result of a limit doing straight substitution yielded $\frac{0}{0}$ we used factoring or conjugation to be able to solve the problem. What happens when neither of those methods prove useful? You become very grateful for the 17th-century French mathematician Guillaume de L'Hôpital. L'Hôpital was the man that derived a method of solving these types of equations, known as indeterminate forms. This method, known as L'Hôpital's Rule, is formally defined below.

Formal Definition: L'Hôpital's Rule

If the limit $\lim_{x \to c} \frac{f(x)}{g(x)}$ results in one of the following forms:

$$\frac{0}{0}, \quad \pm\frac{\infty}{\infty}, \quad 0 * \pm\infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0$$

And $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exits and $g'(x) \neq 0$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Example 1: indeterminate form of $\frac{0}{0}$

Find the limit $\lim_{x \to 0} \frac{e^x - 1}{x}$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x}{1}$$

Using L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = \frac{1}{1} = 1$$

Example 2: indeterminate form of $\frac{\infty}{\infty}$

Find the limit $\lim_{x \to \infty} \frac{x^2}{2^x}$

$$\lim_{x \to \infty} \frac{x^2}{2^x} = \frac{\infty}{\infty}$$

Using L'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{x^2}{2^x} = \lim_{x \to \infty} \frac{2x}{2^x ln2} = \frac{2}{ln2} \lim_{x \to \infty} \frac{x}{2^x} = \frac{\infty}{\infty}$$

Using L'Hôpital's Rule again:

$$\frac{2}{ln2} \lim_{x \to \infty} \frac{x}{2^x} = \frac{2}{ln2} \lim_{x \to \infty} \frac{1}{2^x ln2} = \frac{2}{(ln2)^2} \lim_{x \to \infty} \frac{1}{2^x} = \frac{2}{(ln2)^2} * 0 = 0$$

Example 3: indeterminate form of $\infty - \infty$

Find the limit $\lim_{x\to 2}\left(\frac{4}{x^2-4} - \frac{1}{x-2}\right)$

$$\lim_{x\to 2}\left(\frac{4}{x^2-4} - \frac{1}{x-2}\right) = \infty - \infty$$

Using L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{4}{x^2-4} - \frac{1}{x-2}\right) = \lim_{x\to 2}\frac{4-(x+2)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{2-x}{x^2-4} = \lim_{x\to 2}\frac{-1}{2x} = -\frac{1}{4}$$

Example 4: indeterminate form of $\mathbf{1^\infty}$

Find the limit $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = 1^\infty$$

Let $y = x^{\frac{1}{1-x}}$. Then $\ln y = \ln x^{\frac{1}{1-x}} = \frac{\ln x}{1-x}$

Using L'Hôpital's Rule:

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln x}{1-x} = \lim_{x\to 1}\frac{\frac{1}{x}}{-1} = -\lim_{x\to 1}\frac{1}{x} = -1$$

Therefore

$$\lim_{x\to 1}\ln y = -1$$

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \lim_{x\to 1} y = e^{-1} = \frac{1}{e}$$

Example 5: indeterminate form of $\mathbf{0} * \infty$

Find the limit $\lim_{x\to 0^+}(x * \ln x)$

$$\lim_{x\to 0^+} x * \ln x = 0 * \infty$$

Using L'Hôpital's Rule:

$$\lim_{x\to 0^+} x * \ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

Note the trick that is needed here – what happens if you use L'Hôpital's Rule without making this initial change?