Limits to Infinity (Horizontal Asymptotes)

What happens to a function as it goes further and further out to the left and right? Well, that depends on the function. But half of the answer can be discovered by allowing the independent variable to take on increasingly large, positive values and keeping an eye on the output (the graph) - this investigates what is happening as we go further and further to the right. The other half is discovered by allowing the independent variable to take on increasingly large, negative values and, again, keeping an eye on the output – this investigates what is happening as we go further and further to the left.

Here are some basic facts and some generalizations that will be sufficient to evaluate most "limits to infinity".

Consider the function as an algebraic fraction, and consider the ratio of the leading terms. Let the algebraic expression in the numerator be expressed as *n(x)* and the algebraic expression found in the denominator be expressed as *d(x)*, then

- If the degree of the numerators is lower than the degree of the denominator, then $\lim_{x\to\infty}\frac{n(x)}{d(x)} = 0$. In general, whenever the denominator grows faster than the numerator, the limit will go to zero. Thus, in these cases, as the graph extends far to both the left and the right, the output (i.e., the graph) gets closer and closer to zero.

  Here is a list of functions in order of their rate of growth – quickest to slowest:
  $x!$, ... 4^x, 3^x, 2^x, ... x^4, x^3, x^2, $x\log x$, x, $\log x$, ... , 3, 2, 1

  Examples:

  a. $\lim_{x\to\infty}\frac{5\,x^3-2x^2+7x-13}{12-2x+x^4} = \lim_{x\to\infty}\frac{5x^3}{x^4} = \lim_{x\to\infty}\frac{5}{x} = 0$

  b. $\lim_{x\infty\to}\frac{6x\log x+5x-2}{2^x\mp 2x-14\log x} = \lim_{x\to\infty}\frac{6x\log x}{2^x} = 0$

- If the degree of the numerator is higher than the degree of the denominator, then $\lim_{x\to\infty}\frac{n(x)}{d(x)} = \infty \; or \; -\infty$. In general, whenever the numerator grows faster than the denominator, the limit will go to positive or negative infinity. Thus, in these cases, as the graph extends far to both the left and the right, the output (i.e., the graph) increases or decreases without bound. In these cases, each side needs to be considered separately.

Examples:

a. $\lim_{x\to\infty}\frac{3x^5+x^2-5}{6+x+7x^2} = \lim_{x\to\infty}\frac{3x^5}{7x^2} = \lim_{x\to\infty}\frac{3}{7}x^3 = \infty$

whereas $\lim_{x\to-\infty}\frac{3x^5+x^2-5}{6+x+7x^2} = \lim_{x\to-\infty}\frac{3}{7}x^3 = -\infty$

b. $\lim_{x\to\infty}\frac{x!-\log x+1}{2^x+3x^5-5\log x} = \lim_{x\to\infty}\frac{x!}{2^x} = +\infty$ [Note that $\lim_{x\to-\infty}\frac{x!-\log x+1}{2^x+3x^5-5\log x}$ is meaningless since *x*! is not defined for negative values.]

c. $\lim_{x\to\infty} 2x^4 - 5x + 3$= +∞ (Note that $\lim_{x\to-\infty} 2x^4 - 5x + 3 = +\infty$ as well [why?])

If the degrees are equal, then $\lim_{x\to\infty}\frac{n(x)}{d(x)}$ is equal to the leading coefficient of *n(x)* over the leading coefficient of *d(x)*.

Examples:

a. $\lim_{x\to\infty}\frac{3x^3-2x^2+x-4}{4+2x-5x^3} = \lim_{x\to\infty}\frac{3x^3}{-5x^3} = \lim_{x\to\infty}\frac{3}{-5} = -\frac{3}{5}$

b. $\lim_{x\to\infty}\frac{\frac{1}{2}2^x}{\frac{3}{4}2^x-\log x+4}=\frac{1/2}{3/4} = \frac{2}{3}$