# Limits for Practice

**1.** Which of the following is indeterminate at $x = 1$?

$$\frac{x^2+1}{x-1}, \qquad \frac{x^2-1}{x+2}, \qquad \frac{x^2-1}{\sqrt{x+3}-2}, \qquad \frac{x^2+1}{\sqrt{x+3}-2}$$

**SOLUTION** At $x = 1$, $\frac{x^2-1}{\sqrt{x+3}-2}$ is of the form $\frac{0}{0}$; hence, this function is indeterminate. None of the remaining functions is indeterminate at $x = 1$: $\frac{x^2+1}{x-1}$ and $\frac{x^2+1}{\sqrt{x+3}-2}$ are undefined because the denominator is zero but the numerator is not, while $\frac{x^2-1}{x+2}$ is equal to 0.

**2.** Give counterexamples to show that these statements are false:

**(a)** If $f(c)$ is indeterminate, then the right- and left-hand limits as $x \to c$ are not equal.

**(b)** If $\lim_{x\to c} f(x)$ exists, then $f(c)$ is not indeterminate.

**(c)** If $f(x)$ is undefined at $x = c$, then $f(x)$ has an indeterminate form at $x = c$.

**SOLUTION**

**(a)** Let $f(x) = \frac{x^2-1}{x-1}$. At $x = 1$, $f$ is indeterminate of the form $\frac{0}{0}$ but

$$\lim_{x\to 1-} \frac{x^2-1}{x-1} = \lim_{x\to 1-} (x+1) = 2 = \lim_{x\to 1+} (x+1) = \lim_{x\to 1+} \frac{x^2-1}{x-1}.$$

**(b)** Again, let $f(x) = \frac{x^2-1}{x-1}$. Then

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 2$$

but $f(1)$ is indeterminate of the form $\frac{0}{0}$.

**(c)** Let $f(x) = \frac{1}{x}$. Then $f$ is undefined at $x = 0$ but does not have an indeterminate form at $x = 0$.

**3.** The method for evaluating limits discussed in this section is sometimes called "simplify and plug in." Explain how it actually relies on the property of continuity.

**SOLUTION** If $f$ is continuous at $x = c$, then, by definition, $\lim_{x\to c} f(x) = f(c)$; in other words, the limit of a continuous function at $x = c$ is the value of the function at $x = c$. The "simplify and plug-in" strategy is based on simplifying a function which is indeterminate to a continuous function. Once the simplification has been made, the limit of the remaining continuous function is obtained by evaluation.

# Limits for Practice

## Exercises

*In Exercises 1–4, show that the limit leads to an indeterminate form. Then carry out the two-step procedure: Transform the function algebraically and evaluate using continuity.*

**1.** $\lim_{x \to 6} \frac{x^2 - 36}{x - 6}$

SOLUTION When we substitute $x = 6$ into $\frac{x^2-36}{x-6}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and simplifying, we find

$$\lim_{x \to 6} \frac{x^2 - 36}{x - 6} = \lim_{x \to 6} \frac{(x-6)(x+6)}{x - 6} = \lim_{x \to 6} (x + 6) = 12.$$

**2.** $\lim_{h \to 3} \frac{9 - h^2}{h - 3}$

SOLUTION When we substitute $h = 3$ into $\frac{9-h^2}{h-3}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the denominator and simplifying, we find

$$\lim_{h \to 3} \frac{9 - h^2}{h - 3} = \lim_{h \to 3} \frac{(3-h)(3+h)}{h - 3} = \lim_{h \to 3} -(3 + h) = -6.$$

**3.** $\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}$

SOLUTION When we substitute $x = -1$ into $\frac{x^2+2x+1}{x+1}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and simplifying, we find

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1} = \lim_{x \to -1} \frac{(x+1)^2}{x + 1} = \lim_{x \to -1} (x + 1) = 0.$$

**4.** $\lim_{t \to 9} \frac{2t - 18}{5t - 45}$

SOLUTION When we substitute $t = 9$ into $\frac{2t-18}{5t-45}$, we obtain the indeterminate form $\frac{0}{0}$. Upon dividing out the common factor of $t - 9$ from both the numerator and denominator, we find

$$\lim_{t \to 9} \frac{2t - 18}{5t - 45} = \lim_{t \to 9} \frac{2(t-9)}{5(t-9)} = \lim_{t \to 9} \frac{2}{5} = \frac{2}{5}.$$

# Limits for Practice

*In Exercises 5–34, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).*

**5.** $\lim_{x\to 7} \frac{x-7}{x^2-49}$

**SOLUTION** $\lim_{x\to 7} \frac{x-7}{x^2-49} = \lim_{x\to 7} \frac{x-7}{(x-7)(x+7)} = \lim_{x\to 7} \frac{1}{x+7} = \frac{1}{14}.$

**6.** $\lim_{x\to 8} \frac{x^2-64}{x-9}$

**SOLUTION** $\lim_{x\to 8} \frac{x^2-64}{x-9} = \frac{0}{-1} = 0$

**7.** $\lim_{x\to -2} \frac{x^2+3x+2}{x+2}$

**SOLUTION** $\lim_{x\to -2} \frac{x^2+3x+2}{x+2} = \lim_{x\to -2} \frac{(x+1)(x+2)}{x+2} = \lim_{x\to -2} (x+1) = -1.$

**8.** $\lim_{x\to 8} \frac{x^3-64x}{x-8}$

**SOLUTION** $\lim_{x\to 8} \frac{x^3-64x}{x-8} = \lim_{x\to 8} \frac{x(x-8)(x+8)}{x-8} = \lim_{x\to 8} x(x+8) = 8(16) = 128.$

**9.** $\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-25}$

**SOLUTION** $\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-25} = \lim_{x\to 5} \frac{(x-5)(2x+1)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{2x+1}{x+5} = \frac{11}{10}.$

**10.** $\lim_{h\to 0} \frac{(1+h)^3-1}{h}$

**SOLUTION**

$$\lim_{h\to 0} \frac{(1+h)^3-1}{h} = \lim_{h\to 0} \frac{1+3h+3h^2+h^3-1}{h} = \lim_{h\to 0} \frac{3h+3h^2+h^3}{h}$$
$$= \lim_{h\to 0} (3+3h+h^2) = 3+3(0)+0^2 = 3.$$

**11.** $\lim_{x\to -\frac{1}{2}} \frac{2x+1}{2x^2+3x+1}$

**SOLUTION** $\lim_{x\to -\frac{1}{2}} \frac{2x+1}{2x^2+3x+1} = \lim_{x\to -\frac{1}{2}} \frac{2x+1}{(2x+1)(x+1)} = \lim_{x\to -\frac{1}{2}} \frac{1}{x+1} = 2.$

**12.** $\lim_{x\to 3} \frac{x^2-x}{x^2-9}$

**SOLUTION** As $x \to 3$, the numerator $x^2 - x \to 6$ while the denominator $x^2 - 9 \to 0$; thus, this limit does not exist. Checking the one-sided limits, we find

$$\lim_{x\to 3-} \frac{x^2-x}{x^2-9} = \lim_{x\to 3-} \frac{x(x-1)}{(x-3)(x+3)} = -\infty$$

while

$$\lim_{x\to 3+} \frac{x^2-x}{x^2-9} = \lim_{x\to 3+} \frac{x(x-1)}{(x-3)(x+3)} = \infty.$$

# Limits for Practice

**13.** $\lim_{x\to 2} \frac{3x^2-4x-4}{2x^2-8}$

**SOLUTION** $\lim_{x\to 2} \frac{3x^2-4x-4}{2x^2-8} = \lim_{x\to 2} \frac{(3x+2)(x-2)}{2(x-2)(x+2)} = \lim_{x\to 2} \frac{3x+2}{2(x+2)} = \frac{8}{8} = 1.$

**14.** $\lim_{h\to 0} \frac{(3+h)^3-27}{h}$

**SOLUTION**

$$\lim_{h\to 0} \frac{(3+h)^3-27}{h} = \lim_{h\to 0} \frac{27+27h+9h^2+h^3-27}{h} = \lim_{h\to 0} \frac{27h+9h^2+h^3}{h}$$
$$= \lim_{h\to 0} (27+9h+h^2) = 27+9(0)+0^2 = 27.$$

**15.** $\lim_{t\to 0} \frac{4^{2t}-1}{4^t-1}$

**SOLUTION** $\lim_{t\ to 0} \frac{4^{2t}-1}{4^t-1} = \lim_{t\ to 0} \frac{(4^t-1)(4^t+1)}{4^t-1} = \lim_{t\to 0}(4^t+1) = 2.$

**16.** $\lim_{h\to 4} \frac{(h+2)^2-9h}{h-4}$

**SOLUTION** $\lim_{h\to 4} \frac{(h+2)^2-9h}{h-4} = \lim_{h\to 4} \frac{h^2-5h+4}{h-4} = \lim_{h\to 4} \frac{(h-1)(h-4)}{h-4} = \lim_{h\to 4}(h-1) = 3.$

**17.** $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

**SOLUTION** $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 16} \frac{\sqrt{x}-4}{(\sqrt{x}+4)(\sqrt{x}-4)} = \lim_{x\to 16} \frac{1}{\sqrt{x}+4} = \frac{1}{8}.$

**18.** $\lim_{t\to -2} \frac{2t+4}{12-3t^2}$

**SOLUTION** $\lim_{t\to -2} \frac{2t+4}{12-3t^2} = \lim_{t\to -2} \frac{2(t+2)}{-3(t-2)(t+2)} = \lim_{t\to -2} \frac{2}{-3(t-2)} = \frac{1}{6}.$

**19.** $\lim_{y\to 3} \frac{y^2+y-12}{y^3-10y+3}$

**SOLUTION** $\lim_{y\to 3} \frac{y^2+y-12}{y^3-10y+3} = \lim_{y\to 3} \frac{(y-3)(y+4)}{(y-3)(y^2+3y-1)} = \lim_{y\to 3} \frac{(y+4)}{(y^2+3y-1)} = \frac{7}{17}.$

**20.** $\lim_{h\to 0} \frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h}$

**SOLUTION**

$$\lim_{h\to 0} \frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h} = \lim_{h\to 0} \frac{\frac{4-(h+2)^2}{4(h+2)^2}}{h} = \lim_{h\to 0} \frac{\frac{4-(h^2+4h+4)}{4(h+2)^2}}{h} = \lim_{h\to 0} \frac{\frac{-h^2-4h}{4(h+2)^2}}{h}$$
$$= \lim_{h\to 0} \frac{h\frac{-h-4}{4(h+2)^2}}{h} = \lim_{h\to 0} \frac{-h-4}{4(h+2)^2} = \frac{-4}{16} = -\frac{1}{4}.$$

**21.** $\lim_{h\to 0} \frac{\sqrt{2+h}-2}{h}$

**SOLUTION** $\lim_{h\to 0} \frac{\sqrt{h+2}-2}{h}$ does not exist.

- As $h\to 0+$, we have $\frac{\sqrt{h+2}-2}{h} = \frac{(\sqrt{h+2}-2)(\sqrt{h+2}+2)}{h(\sqrt{h+2}+2)} = \frac{h-2}{h(\sqrt{h+2}+2)} \to -\infty.$
- As $h\to 0-$, we have $\frac{\sqrt{h+2}-2}{h} = \frac{(\sqrt{h+2}-2)(\sqrt{h+2}+2)}{h(\sqrt{h+2}+2)} = \frac{h-2}{h(\sqrt{h+2}+2)} \to \infty.$

**22.** $\lim_{x\to 8} \frac{\sqrt{x-4}-2}{x-8}$

**22.** $\lim_{x\to 8} \frac{\sqrt{x-4}-2}{x-8}$

**SOLUTION**

$$\lim_{x\to 8} \frac{\sqrt{x-4}-2}{x-8} = \lim_{x\to 8} \frac{(\sqrt{x-4}-2)(\sqrt{x-4}+2)}{(x-8)(\sqrt{x-4}+2)} = \lim_{x\to 8} \frac{x-4-4}{(x-8)(\sqrt{x-4}+2)}$$
$$= \lim_{x\to 8} \frac{1}{\sqrt{x-4}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

**23.** $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-\sqrt{8-x}}$

**SOLUTION**

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-\sqrt{8-x}} = \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{(\sqrt{x}-\sqrt{8-x})(\sqrt{x}+\sqrt{8-x})} = \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-(8-x)}$$
$$= \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2x-8} = \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2(x-4)}$$
$$= \lim_{x\to 4} \frac{(\sqrt{x}+\sqrt{8-x})}{2} = \frac{\sqrt{4}+\sqrt{4}}{2} = 2.$$

**24.** $\lim_{x\to 4} \frac{\sqrt{5-x}-1}{2-\sqrt{x}}$

**SOLUTION**

$$\lim_{x\to 4} \frac{\sqrt{5-x}-1}{2-\sqrt{x}} = \lim_{x\to 4} \left(\frac{\sqrt{5-x}-1}{2-\sqrt{x}} \cdot \frac{\sqrt{5-x}+1}{\sqrt{5-x}+1}\right) = \lim_{x\to 4} \frac{4-x}{(2-\sqrt{x})(\sqrt{5-x}+1)}$$
$$= \lim_{x\to 4} \frac{(2-\sqrt{x})(2+\sqrt{x})}{(2-\sqrt{x})(\sqrt{5-x}+1)} = \lim_{x\to 4} \frac{2+\sqrt{x}}{\sqrt{5-x}+1} = 2.$$

**25.** $\lim_{x\to 4} \left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right)$

**SOLUTION** $\lim_{x\to 4} \left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right) = \lim_{x\to 4} \frac{\sqrt{x}+2-4}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{1}{4}.$

**26.** $\lim_{x\to 0+} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2+x}}\right)$

**SOLUTION**

$$\lim_{x\to 0+} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2+x}}\right) = \lim_{x\to 0+} \frac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}} = \lim_{x\to 0+} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{\sqrt{x}\sqrt{x+1}(\sqrt{x+1}+1)}$$
$$= \lim_{x\to 0+} \frac{x}{\sqrt{x}\sqrt{x+1}(\sqrt{x+1}+1)} = \lim_{x\to 0+} \frac{\sqrt{x}}{\sqrt{x+1}(\sqrt{x+1}+1)} = 0.$$

**27.** $\lim_{x\to 0} \frac{\cot x}{\csc x}$

**SOLUTION** $\lim_{x\to 0} \frac{\cot x}{\csc x} = \lim_{x\to 0} \frac{\cos x}{\sin x} \cdot \sin x = \cos 0 = 1.$

**28.** $\lim_{\theta\to \frac{\pi}{2}} \frac{\cot\theta}{\csc\theta}$

**SOLUTION** $\lim_{\theta\to \frac{\pi}{2}} \frac{\cot\theta}{\csc\theta} = \lim_{\theta\to \frac{\pi}{2}} \frac{\cos\theta}{\sin\theta} \cdot \sin\theta = \cos\frac{\pi}{2} = 0.$

**29.** $\lim_{t\to 2} \frac{2^{2t}+2^t-20}{2^t-4}$

**SOLUTION** $\lim_{t\to 2} \frac{2^{2t}+2^t-20}{2^t-4} = \lim_{t\to 2} \frac{(2^t+5)(2^t-4)}{2^t-4} = \lim_{t\to 2}(2^t+5) = 9.$

**30.** $\lim_{x\to 1} \left(\frac{1}{1-x} - \frac{2}{1-x^2}\right)$

**30.** $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)$

**SOLUTION** $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)=\lim_{x\to 1}\frac{(1+x)-2}{(1-x)(1+x)}=\lim_{x\to 1}\frac{-1}{1+x}=-\frac{1}{2}.$

**31.** $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}$

**SOLUTION** $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}\cdot\frac{\cos x}{\cos x}=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)\cos x}{\sin x-\cos x}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}.$

**32.** $\lim_{\theta\to\frac{\pi}{2}}\left(\sec\theta-\tan\theta\right)$

**SOLUTION**

$$\lim_{\theta\to\frac{\pi}{2}}\left(\sec\theta-\tan\theta\right)=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{\cos\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin^2\theta}{\cos\theta\,(1+\sin\theta)}=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{1+\sin\theta}=\frac{0}{2}=0.$$

**33.** $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)$

**SOLUTION** $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)=\lim_{\theta\to\frac{\pi}{4}}\frac{(\tan\theta+1)-2}{(\tan\theta+1)(\tan\theta-1)}=\lim_{\theta\to\frac{\pi}{4}}\frac{1}{\tan\theta+1}=\frac{1}{2}.$

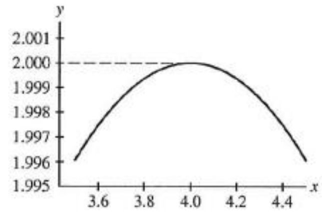
**34.** $\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}$

**SOLUTION**

$$\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\frac{(2\cos x-1)\,(\cos x+2)}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\cos x+2=\cos\frac{\pi}{3}+2=\frac{5}{2}.$$

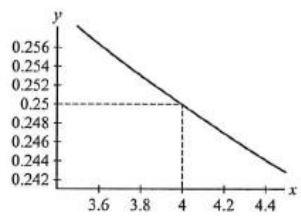
**35.** GU Use a plot of $f(x) = \dfrac{x-4}{\sqrt{x}-\sqrt{8-x}}$ to estimate $\lim_{x \to 4} f(x)$ to two decimal places. Compare with the answer obtained algebraically in Exercise 23.

**SOLUTION** Let $f(x) = \frac{x-4}{\sqrt{x}-\sqrt{8-x}}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x \to 4} f(x) \approx 2.00$; to two decimal places, this matches the value of 2 obtained in Exercise 23.




**36.** GU Use a plot of $f(x) = \dfrac{1}{\sqrt{x}-2} - \dfrac{4}{x-4}$ to estimate $\lim_{x \to 4} f(x)$ numerically. Compare with the answer obtained algebraically in Exercise 25.

**SOLUTION** Let $f(x) = \frac{1}{\sqrt{x}-2} - \frac{4}{x-4}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x \to 4} f(x) \approx 0.25$; to two decimal places, this matches the value of $\frac{1}{4}$ obtained in Exercise 25.



# Limits for Practice

*In Exercises 37–42, evaluate using the identity*

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

**37.** $\lim_{x\to 2} \frac{x^3 - 8}{x - 2}$

**SOLUTION** $\lim_{x\to 2} \frac{x^3 - 8}{x - 2} = \lim_{x\to 2} \frac{(x-2)\left(x^2 + 2x + 4\right)}{x - 2} = \lim_{x\to 2} \left(x^2 + 2x + 4\right) = 12.$

**38.** $\lim_{x\to 3} \frac{x^3 - 27}{x^2 - 9}$

**SOLUTION** $\lim_{x\to 3} \frac{x^3 - 27}{x^2 - 9} = \lim_{x\to 3} \frac{(x-3)\left(x^2 + 3x + 9\right)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{\left(x^2 + 3x + 9\right)}{x + 3} = \frac{27}{6} = \frac{9}{2}.$

**39.** $\lim_{x\to 1} \frac{x^2 - 5x + 4}{x^3 - 1}$

**SOLUTION** $\lim_{x\to 1} \frac{x^2 - 5x + 4}{x^3 - 1} = \lim_{x\to 1} \frac{(x-1)(x-4)}{(x-1)\left(x^2 + x + 1\right)} = \lim_{x\to 1} \frac{x - 4}{x^2 + x + 1} = -1.$

**40.** $\lim_{x\to -2} \frac{x^3 + 8}{x^2 + 6x + 8}$

**SOLUTION** $\lim_{x\to -2} \frac{x^3 + 8}{x^2 + 6x + 8} = \lim_{x\to -2} \frac{(x+2)(x^2 - 2x + 4)}{(x+2)(x+4)} = \lim_{x\to -2} \frac{(x^2 - 2x + 4)}{x + 4} = \frac{12}{2} = 6.$

**41.** $\lim_{x\to 1} \frac{x^4 - 1}{x^3 - 1}$

**SOLUTION**

$$\lim_{x\to 1} \frac{x^4 - 1}{x^3 - 1} = \lim_{x\to 1} \frac{(x^2 - 1)(x^2 + 1)}{(x-1)(x^2 + x + 1)} = \lim_{x\to 1} \frac{(x-1)(x+1)(x^2 + 1)}{(x-1)(x^2 + x + 1)} = \lim_{x\to 1} \frac{(x+1)(x^2 + 1)}{(x^2 + x + 1)} = \frac{4}{3}.$$

**42.** $\lim_{x\to 27} \frac{x - 27}{x^{1/3} - 3}$

**SOLUTION** $\lim_{x\to 27} \frac{x - 27}{x^{1/3} - 3} = \lim_{x\to 27} \frac{(x^{1/3} - 3)(x^{2/3} + 3x^{1/3} + 9)}{x^{1/3} - 3} = \lim_{x\to 27} (x^{2/3} + 3x^{1/3} + 9) = 27$

**43.** Evaluate $\lim_{h\to 0} \frac{\sqrt[4]{1+h}-1}{h}$. *Hint:* Set $x = \sqrt[4]{1+h}$ and rewrite as a limit as $x \to 1$.

**SOLUTION** Let $x = \sqrt[4]{1+h}$. Then $h = x^4 - 1 = (x-1)(x+1)(x^2+1)$, $x \to 1$ as $h \to 0$ and

$$\lim_{h\to 0} \frac{\sqrt[4]{1+h}-1}{h} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)(x^2+1)} = \lim_{x\to 1} \frac{1}{(x+1)(x^2+1)} = \frac{1}{4}.$$

**44.** Evaluate $\lim_{h\to 0} \frac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1}$. *Hint:* Set $x = \sqrt[6]{1+h}$ and rewrite as a limit as $x \to 1$.

**SOLUTION** Let $x = \sqrt[6]{1+h}$. Then $\sqrt[3]{1+h}-1 = x^2 - 1 = (x-1)(x+1)$, $\sqrt{1+h}-1 = x^3 - 1 = (x-1)(x^2+x+1)$, $x \to 1$ as $h \to 0$ and

$$\lim_{h\to 0} \frac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}.$$

*In Exercises 45–54, evaluate in terms of the constant $a$.*

**45.** $\lim_{x\to 0} (2a + x)$

**SOLUTION** $\lim_{x\to 0} (2a + x) = 2a$.

**46.** $\lim_{h\to -2} (4ah + 7a)$

**SOLUTION** $\lim_{h\to -2} (4ah + 7a) = -a$.

**47.** $\lim_{t\to -1} (4t - 2at + 3a)$

**SOLUTION** $\lim_{t\to -1} (4t - 2at + 3a) = -4 + 5a$.

# Limits for Practice

**48.** $\lim_{h\to 0} \frac{(3a+h)^2 - 9a^2}{h}$

**SOLUTION** $\lim_{h\to 0} \frac{(3a+h)^2 - 9a^2}{h} = \lim_{h\to 0} \frac{6ah + h^2}{h} = \lim_{h\to 0} (6a + h) = 6a.$

**49.** $\lim_{h\to 0} \frac{2(a+h)^2 - 2a^2}{h}$

**SOLUTION** $\lim_{h\to 0} \frac{2(a+h)^2 - 2a^2}{h} = \lim_{h\to 0} \frac{4ha + 2h^2}{h} = \lim_{h\to 0} (4a + 2h) = 4a.$

**50.** $\lim_{x\to a} \frac{(x+a)^2 - 4x^2}{x-a}$

**SOLUTION**

$$\lim_{x\to a} \frac{(x+a)^2 - 4x^2}{x-a} = \lim_{x\to a} \frac{(x^2 + 2ax + a^2) - 4x^2}{x-a} = \lim_{x\to a} \frac{-3x^2 + 2ax + a^2}{x-a}$$
$$= \lim_{x\to a} \frac{(a-x)(a+3x)}{x-a} = \lim_{x\to a} (-(a+3x)) = -4a.$$

**51.** $\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a}$

**SOLUTION** $\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a} = \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{\left(\sqrt{x} - \sqrt{a}\right)\left(\sqrt{x} + \sqrt{a}\right)} = \lim_{x\to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}.$

**52.** $\lim_{h\to 0} \frac{\sqrt{a+2h} - \sqrt{a}}{h}$

**SOLUTION**

$$\lim_{h\to 0} \frac{\sqrt{a+2h} - \sqrt{a}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{a+2h} - \sqrt{a}\right)\left(\sqrt{a+2h} + \sqrt{a}\right)}{h\left(\sqrt{a+2h} + \sqrt{a}\right)}$$
$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{a+2h} + \sqrt{a}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{a+2h} + \sqrt{a}} = \frac{1}{\sqrt{a}}.$$

**53.** $\lim_{x\to 0} \frac{(x+a)^3 - a^3}{x}$

**SOLUTION** $\lim_{x\to 0} \frac{(x+a)^3 - a^3}{x} = \lim_{x\to 0} \frac{x^3 + 3x^2a + 3xa^2 + a^3 - a^3}{x} = \lim_{x\to 0} (x^2 + 3xa + 3a^2) = 3a^2.$

**54.** $\lim_{h\to a} \frac{\frac{1}{h} - \frac{1}{a}}{h-a}$

**SOLUTION** $\lim_{h\to a} \frac{\frac{1}{h} - \frac{1}{a}}{h-a} = \lim_{h\to a} \frac{\frac{a-h}{ah}}{h-a} = \lim_{h\to a} \frac{a-h}{ah} \frac{1}{h-a} = \lim_{h\to a} \frac{-1}{ah} = -\frac{1}{a^2}$

## *Further Insights and Challenges*

*In Exercises 55–58, find all values of c such that the limit exists.*

**55.** $\lim_{x \to c} \frac{x^2 - 5x - 6}{x - c}$

**SOLUTION** $\lim_{x \to c} \frac{x^2 - 5x - 6}{x - c}$ will exist provided that $x - c$ is a factor of the numerator. (Otherwise there will be an infinite discontinuity at $x = c$.) Since $x^2 - 5x - 6 = (x + 1)(x - 6)$, this occurs for $c = -1$ and $c = 6$.

**56.** $\lim_{x \to 1} \frac{x^2 + 3x + c}{x - 1}$

**SOLUTION** $\lim_{x \to 1} \frac{x^2 + 3x + c}{x - 1}$ exists as long as $(x - 1)$ is a factor of $x^2 + 3x + c$. If $x^2 + 3x + c = (x - 1)(x + q)$, then $q - 1 = 3$ and $-q = c$. Hence $q = 4$ and $c = -4$.

**57.** $\lim_{x \to 1} \left( \frac{1}{x - 1} - \frac{c}{x^3 - 1} \right)$

**SOLUTION** Simplifying, we find

$$\frac{1}{x - 1} - \frac{c}{x^3 - 1} = \frac{x^2 + x + 1 - c}{(x - 1)(x^2 + x + 1)}.$$

In order for the limit to exist as $x \to 1$, the numerator must evaluate to 0 at $x = 1$. Thus, we must have $3 - c = 0$, which implies $c = 3$.

# Limits for Practice

## Trigonometric Limits

### Preliminary Questions

**1.** Assume that $-x^4 \le f(x) \le x^2$. What is $\lim_{x\to 0} f(x)$? Is there enough information to evaluate $\lim_{x\to \frac{1}{2}} f(x)$? Explain.

SOLUTION Since $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^2 = 0$, the squeeze theorem guarantees that $\lim_{x\to 0} f(x) = 0$. Since $\lim_{x\to \frac{1}{2}} -x^4 = -\frac{1}{16} \neq \frac{1}{4} = \lim_{x\to \frac{1}{2}} x^2$, we do not have enough information to determine $\lim_{x\to \frac{1}{2}} f(x)$.

**2.** State the Squeeze Theorem carefully.

SOLUTION Assume that for $x \neq c$ (in some open interval containing $c$),

$$l(x) \le f(x) \le u(x)$$

and that $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$. Then $\lim_{x\to c} f(x)$ exists and

$$\lim_{x\to c} f(x) = L.$$

**3.** If you want to evaluate $\lim_{h\to 0} \frac{\sin 5h}{3h}$, it is a good idea to rewrite the limit in terms of the variable (choose one):

**(a)** $\theta = 5h$ **(b)** $\theta = 3h$ **(c)** $\theta = \frac{5h}{3}$

SOLUTION To match the given limit to the pattern of

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta},$$

it is best to substitute for the argument of the sine function; thus, rewrite the limit in terms of **(a):** $\theta = 5h$.