The Chain Rule … Fact

Chain Rule

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1. If we define $F(x) = (f \circ g)(x)$ then the derivative of $F(x)$ is,

$$F'(x) = f'(g(x)) \; g'(x)$$

2. If we have $y = f(u)$ and $u = g(x)$ then the derivative of y is,

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

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Example 1 Use the Chain Rule to differentiate $R(z) = \sqrt{5z - 8}$.

In general, we don't really do all the composition stuff in using the Chain Rule. That can get a little complicated and in fact obscures the fact that there is a quick and easy way of remembering the chain rule that doesn't require us to think in terms of function composition.

Let's take the function from the previous example and rewrite it slightly.

$$R(z) = \underbrace{5z - 8}_{\text{inside function}} \quad \underbrace{(\quad)}_{\substack{\text{outside}\\ \text{function}}}^{\frac{1}{2}}$$

This function has an "inside function" and an "outside function". The outside function is the square root or the exponent of $\frac{1}{2}$ depending on how you want to think of it and the inside function is the stuff that we're taking the square root of or raising to the $\frac{1}{2}$, again depending on how you want to look at it.

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The derivative is then,

$$R'(z) = \overbrace{\frac{1}{2}\underbrace{(5z-8)}_{\substack{\text{inside function}\\ \text{left alone}}}^{-\frac{1}{2}}}^{\substack{\text{derivative of}\\ \text{outside function}}}\underbrace{(5)}_{\substack{\text{derivative of}\\ \text{inside function}}}$$

In general, this is how we think of the chain rule. We identify the "inside function" and the "outside function". We then differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of}\\ \text{outside function}}}\underbrace{(g(x))}_{\substack{\text{inside function}\\ \text{left alone}}}\underbrace{g'(x)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

We can always identify the "outside function" in the examples below by asking ourselves how we would evaluate the function. For instance in the $R(z)$ case if we were to ask ourselves what $R(2)$ is we would first evaluate the stuff under the radical and then finally take the square root of this result. The square root is the last operation that we perform in the evaluation and this is also the outside function. The outside function will always be the last operation you would perform if you were going to evaluate the function.

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We've already identified the two functions that we needed for the composition, but let's write them back down anyway and take their derivatives.

$$f(z) = \sqrt{z} \qquad\qquad g(z) = 5z - 8$$
$$f'(z) = \frac{1}{2\sqrt{z}} \qquad\qquad g'(z) = 5$$

So, using the chain rule we get,

$$\begin{aligned} R'(z) &= f'(g(z))\ g'(z) \\ &= f'(5z-8)\ g'(z) \\ &= \frac{1}{2}(5z-8)^{-\frac{1}{2}}\ (5) \\ &= \frac{1}{2\sqrt{5z-8}}\ (5) \\ &= \frac{5}{2\sqrt{5z-8}} \end{aligned}$$