Derivatives of Exponential Functions and Logarithmic Functions
… Set 2

Derivatives of exponential and logarithmic functions

If you are not familiar with exponential and logarithmic functions you may wish to consult the booklet *Exponents and Logarithms* which is available from the Mathematics Learning Centre.

You may have seen that there are two notations popularly used for natural logarithms, $\log_e$ and ln. These are just two different ways of writing exactly the same thing, so that $\log_e x \equiv \ln x$. In this booklet we will use both these notations.

The basic results are:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}.$$

Example

Differentiate $\log_e(x^2+3x+1)$.

Solution

We solve this by using the chain rule and our knowledge of the derivative of $\log_e x$.

$$\begin{aligned}
\frac{d}{dx}\log_e(x^2+3x+1) &= \frac{d}{dx}(\log_e u) \qquad (\text{where } u = x^2+3x+1) \\
&= \frac{d}{du}(\log_e u) \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\
&= \frac{1}{u} \times \frac{du}{dx} \\
&= \frac{1}{x^2+3x+1} \times \frac{d}{dx}(x^2+3x+1) \\
&= \frac{1}{x^2+3x+1} \times (2x+3) \\
&= \frac{2x+3}{x^2+3x+1}.
\end{aligned}$$

Example

Find $\frac{d}{dx}(e^{3x^2})$.

Solution

This is an application of the chain rule together with our knowledge of the derivative of e^x.

$$\begin{aligned}
\frac{d}{dx}(e^{3x^2}) &= \frac{de^u}{dx} \qquad \text{where } u = 3x^2 \\
&= \frac{de^u}{du} \times \frac{du}{dx} \qquad \text{by the chain rule} \\
&= e^u \times \frac{du}{dx} \\
&= e^{3x^2} \times \frac{d}{dx}(3x^2) \\
&= 6xe^{3x^2}.
\end{aligned}$$

Example

Find $\frac{d}{dx}(e^{x^3+2x})$.

Solution

Again, we use our knowledge of the derivative of e^x together with the chain rule.

$$\begin{aligned} \frac{d}{dx}(e^{x^3+2x}) &= \frac{de^u}{dx} \qquad \text{(where } u = x^3+2x\text{)} \\ &= e^u \times \frac{du}{dx} \qquad \text{(by the chain rule)} \\ &= e^{x^3+2x} \times \frac{d}{dx}(x^3+2x) \\ &= (3x^2+2) \times e^{x^3+2x}. \end{aligned}$$

Example

Differentiate $\ln(2x^3 + 5x^2 - 3)$.

Solution

We solve this by using the chain rule and our knowledge of the derivative of $\ln x$.

$$\begin{aligned}
\frac{d}{dx}\ln(2x^3+5x^2-3) &= \frac{d\ln u}{dx} \qquad \text{(where } u = (2x^3+5x^2-3)\text{)} \\
&= \frac{d\ln u}{du} \times \frac{du}{dx} \qquad \text{(by the chain rule)} \\
&= \frac{1}{u} \times \frac{du}{dx} \\
&= \frac{1}{2x^3+5x^2-3} \times \frac{d}{dx}(2x^3+5x^2-3) \\
&= \frac{1}{2x^3+5x^2-3} \times (6x^2+10x) \\
&= \frac{6x^2+10x}{2x^3+5x^2-3}.
\end{aligned}$$

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There are two shortcuts to differentiating functions involving exponents and logarithms. The four examples above gave

$$\begin{aligned}
\frac{d}{dx}(\log_e(x^2+3x+1)) &= \frac{2x+3}{x^2+3x+1} \\
\frac{d}{dx}(e^{3x^2}) &= 6xe^{3x^2} \\
\frac{d}{dx}(e^{x^3+2x}) &= (3x^2+2)e^{3x^2} \\
\frac{d}{dx}(\log_e(2x^3+5x^2-3)) &= \frac{6x^2+10x}{2x^3+5x^2-3}.
\end{aligned}$$

These examples suggest the general rules

$$\begin{aligned}
\frac{d}{dx}(e^{f(x)}) &= f'(x)e^{f(x)} \\
\frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)}.
\end{aligned}$$

These rules arise from the chain rule and the fact that $\frac{de^x}{dx} = e^x$ and $\frac{d\ln x}{dx} = \frac{1}{x}$. They can speed up the process of differentiation but it is not necessary that you remember them. If you forget, just use the chain rule as in the examples above.

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Exercise 1

Differentiate the following functions.

a. $f(x) = \ln(2x^3)$ **b.** $f(x) = e^{x^7}$ **c.** $f(x) = \ln(11x^7)$

d. $f(x) = e^{x^2+x^3}$ **e.** $f(x) = \log_e(7x^{-2})$ **f.** $f(x) = e^{-x}$

g. $f(x) = \ln(e^x + x^3)$ **h.** $f(x) = \ln(e^x x^3)$ **i.** $f(x) = \ln\left(\dfrac{x^2+1}{x^3-x}\right)$

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Answers

Solutions to Exercise 1

a. $f'(x) = \dfrac{6x^2}{2x^3} = \dfrac{3}{x}$

Alternatively write $f(x) = \ln 2 + 3\ln x$ so that $f'(x) = 3\dfrac{1}{x}$.

b. $f'(x) = 7x^6 e^{x^7}$

c. $f'(x) = \frac{7}{x}$

d. $f'(x) = (2x + 3x^2)e^{x^2+x^3}$

e. Write $f(x) = \log_e 7 - 2\log_e x$ so that $f'(x) = -\frac{2}{x}$.

f. $f'(x) = -e^{-x}$

g. $f'(x) = \dfrac{e^x + 3x^2}{e^x + x^3}$

h. Write $f(x) = \ln e^x + \frac{3}{\ln x}$ so that $f'(x) = 1 + \dfrac{3}{x}$.

i. Write $f(x) = \ln(x^2+1) - \ln(x^3 - x)$ so that $f'(x) = \dfrac{2x}{x^2+1} - \dfrac{3x^2-1}{x^3-x}$.