Derivatives of Exponential and Logarithmic Functions. Logarithmic Differentiation

Derivative of exponential functions. The natural exponential function can be considered as "the easiest function in Calculus courses" since

> the derivative of e^x is e^x.

General Exponential Function a^x. Assuming the formula for e^x, you can obtain the formula for the derivative of any other base $a > 0$ by noting that $y = a^x$ is equal to $e^{\ln a^x} = e^{x \ln a}$. Use chain rule and the formula for derivative of e^x to obtain that $y' = e^{x \ln a} \ln a = a^x \ln a$. Thus

> the derivative of a^x is $a^x \ln a$.

Derivative of the inverse function. If $f(x)$ is a one-to-one function (i.e. the graph of $f(x)$ passes the horizontal line test), then $f(x)$ has the inverse function $f^{-1}(x)$. Recall that f and f^{-1} are related by the following formulas

$$y = f^{-1}(x) \iff x = f(y).$$

Also, recall that the graphs of $f^{-1}(x)$ and $f(x)$ are symmetrical with respect to line $y = x$.

Some pairs of inverse functions you encountered before are given in the following table where n is a positive integer and a is a positive real number.

| f	x^2	x^n	e^x	a^x
f^{-1}	$\sqrt{x}$	$\sqrt[n]{x}$	$\ln x$	$\log_a x$

With $y = f^{-1}(x)$, $\frac{dy}{dx}$ denotes the derivative of f^{-1} and since $x = f(y)$, $\frac{dx}{dy}$ denotes the derivative of f. Since the reciprocal of $\frac{dy}{dx}$ is $\frac{dx}{dy}$ we have that

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}.$$

Thus, the derivative of the inverse function of f is reciprocal of the derivative of f.

Graphically, this rule means that

> The slope of the tangent to $f^{-1}(x)$ at point (b, a) is reciprocal to the slope of the tangent to $f(x)$ at point (a, b).
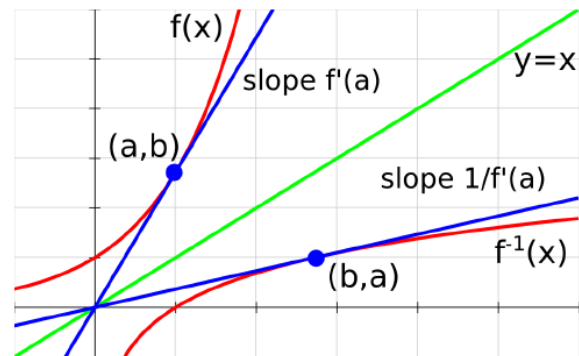
Derivatives of Exponential Functions and Logarithmic Functions
## ... Set 3

Logarithmic function and their derivatives.

Recall that the function $\log_a x$ is the inverse function of a^x: thus $\log_a x = y \Leftrightarrow a^y = x$.

If $a = e$, the notation $\ln x$ is short for $\log_e x$ and the function $\ln x$ is called the **natural logarithm**.

The derivative of $y = \ln x$ can be obtained from derivative of the inverse function $x = e^y$. Note that the derivative x' of $x = e^y$ is $x' = e^y = x$ and consider the reciprocal:

$$y = \ln x \;\Rightarrow\; y' = \frac{1}{x'} = \frac{1}{e^y} = \frac{1}{x}.$$
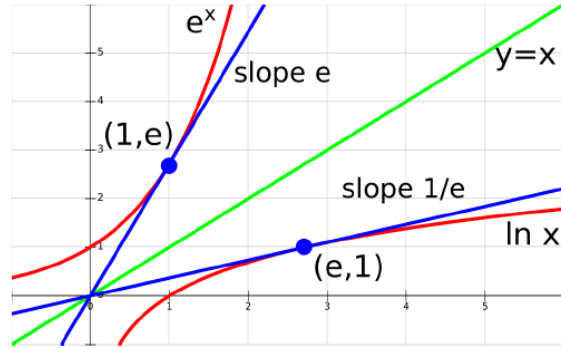
The derivative of logarithmic function of any base can be obtained converting $\log_a$ to $\ln$ as $y = \log_a x = \frac{\ln x}{\ln a} = \ln x \frac{1}{\ln a}$ and using the formula for derivative of $\ln x$. So we have

$$\frac{d}{dx} \log_a x = \frac{1}{x} \frac{1}{\ln a} = \frac{1}{x \ln a}.$$

The derivative of $\ln x$ is $\frac{1}{x}$ and the derivative of $\log_a x$ is $\frac{1}{x \ln a}$.

To summarize,

y	e^x	a^x	$\ln x$	$\log_a x$
y'	e^x	$a^x \ln a$	$\frac{1}{x}$	$\frac{1}{x \ln a}$

Besides two logarithm rules we used above, we recall another two rules which can also be useful.

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^r) = r \log_a x \qquad \log_a x = \frac{\ln x}{\ln a}$$

Logarithmic Differentiation.

Assume that the function has the form $y = f(x)^{g(x)}$ where both f and g are non-constant functions. Although this function is not implicit, it does not fall under any of the forms for which we developed differentiation formulas so far. This is because of the following.

- In order to use the power rule, the exponent needs to be constant.

- In order to use the exponential function differentiation formula, the base needs to be constant.

Thus, no differentiation rule covers the case $y = f(x)^{g(x)}$. These functions sill can be differentiated by using the method known as the **logarithmic differentiation**.

Derivatives of Exponential Functions and Logarithmic Functions
... Set 3

To differentiate a function of the form $y = f(x)^{g(x)}$ follow the steps of the logarithmic differentiation

1. Take ln of both sides of the equation $y = f(x)^{g(x)}$.

2. Rewrite the right side $\ln f(x)^{g(x)}$ as $g(x) \cdot \ln(f(x))$.

3. Differentiate both sides.

4. Solve the resulting equation for y'.

Example 1. Find the derivative of $y = x^x$.

   Solution. Follow the steps of the logarithmic differentiation.

1. First take ln of each side to get $\ln y = \ln x^x$.

2. Rewrite the right side as $x \ln x$ to get $\ln y = x \ln x$.

3. Then differentiate both sides. Use the chain rule for the left side noting that the derivative of the inner function y is y'. Use the product rule for the right side. Obtain $\frac{1}{y}y' = \ln x + \frac{1}{x}x$.

4. Multiply both sides with y to solve for y' and get $y' = (\ln x + 1)y$. Finally, recall that $y = x^x$ to get the derivative solely in terms of x as

$$y' = (\ln x + 1)x^x.$$

Derivatives of Exponential Functions and Logarithmic Functions
... Set 3

Practice Problems:

1. Find the derivatives of the following functions. In parts (g), (h) and (p) a and b are arbitrary constants.

(a) $y = (x^2 + 1)e^{3x}$

(b) $y = e^{x^2+3x}$

(c) $y = \frac{e^{2x}+e^{-2x}}{x^2}$

(d) $y = 3^{x^2+3x}$

(e) $y = x\, 5^{3x}$

(f) $y = \frac{3^x-3^{-x}}{2}$

(g) $y = xe^{ax^2+1}$

(h) $y = \sqrt{1+ae^x}$

(i) $y = (2x + e^{x^2})^4$

(j) $y = \ln(x^2 + 2x)$

(k) $y = \log_2(3x + 4)$

(l) $y = x\ln(x^2 + 1)$

(m) $y = \log_3(x^2 + 5)$

(n) $y = \frac{x\ln x}{x^2+1}$

(o) $y = \ln(x + 5e^{3x})$

(p) $y = ax\ln(x^2 + b^2)$

(q) $y = (3x)^{5x}$

(r) $y = x^{\ln x}$

(s) $y = (\ln x)^x$

(t) $y = (3x + 2)^{2x-1}$

Derivatives of Exponential Functions and Logarithmic Functions … Set 3

Answers

1. (a) Using product rule with $f(x) = x^2 + 1$ and $g(x) = e^{3x}$ and chain for derivative of $g(x)$ obtain $y' = 2xe^{3x} + 3e^{3x}(x^2 + 1)$.

   (b) Use the chain rule. $y' = e^{x^2+3x}(2x + 3)$

   (c) The quotient rule with $f(x) = e^{2x} + e^{-2x}$ and $g(x) = x^2$ and the chain for $f'(x) = 2e^{2x} - 2e^{-2x}$ produces $y' = \frac{(2e^{2x}-2e^{-2x})x^2-2x(e^{2x}+e^{-2x})}{x^4} = \frac{2x((x-1)e^{2x}-(x+1)e^{-2x})}{x^4} = \frac{2((x-1)e^{2x}-(x+1)e^{-2x})}{x^3}$.

   (d) Use the chain rule. $y' = 3^{x^2+3x}\ln 3(2x + 3)$

   (e) Use the product rule with $f(x) = x$ and $g(x) = 5^{3x}$ and the chain for $g'(x) = 5^{3x}\ln 3(3)$ so that $y' = 5^{3x} + 3x\ln 5\, 5^{3x}$.

   (f) $y = \frac{1}{2}(3^x - 3^{-x})$. The derivative of 3^x is $3^x\ln 3$ and, using the chain rule with inner function $-x$, the derivative of 3^{-x} is $3^{-x}\ln 3(-1) = -3^{-x}\ln 3$. Thus $y' = \frac{1}{2}(3^x\ln 3 + 3^{-x}\ln 3) = \frac{\ln 3}{2}(3^x + 3^{-x})$.

   (g) Use the product rule with $f = x$ and $g = e^{ax^2+1}$. Use the chain rule to find the derivative g' as $e^{ax^2+1}a\,2x$. Thus $y' = e^{ax^2+1} + 2ax^2e^{ax^2+1}$.

   (h) Use the chain rule. $y' = \frac{ae^x}{2\sqrt{1+ae^x}}$.

   (i) The chain rule with inner $2x + e^{x^2}$ and another chain rule for derivative of e^{x^2} produces $y' = 4(2x + e^{x^2})^3 \cdot (2 + e^{x^2}2x) = 8(1 + xe^{x^2})(2x + e^{x^2})^3$.

   (j) Chain rule: $y' = \frac{1}{x^2+2x}(2x + 2) = \frac{2x+2}{x^2+2x}$

   (k) Chain rule: $y' = \frac{3}{\ln 2(3x+4)}$

   (l) Chain and product $y' = \ln(x^2 + 1) + \frac{2x^2}{x^2+1}$

   (m) Chain rule: $y' = \frac{2x}{\ln 3(x^2+5)}$

   (n) Product and quotient: $y' = \frac{(\ln x+1)(x^2+1)-2x^2\ln x}{(x^2+1)^2}$

   (o) Chain rule twice: $y' = \frac{1}{x+5e^{3x}}(1 + 5e^{3x}3) = \frac{1+15e^{3x}}{x+5e^{3x}}$

   (p) Product and chain: $y' = a\ln(x^2 + b^2) + \frac{2x}{x^2+b^2}ax = a\ln(x^2 + b^2) + \frac{2ax^2}{x^2+b^2}$.

   (q) Use logarithmic differentiation $\ln y = \ln(3x)^{5x} = 5x\ln(3x) \Rightarrow \frac{1}{y}y' = 5\ln(3x) + \frac{3}{3x}5x \Rightarrow y' = (5\ln(3x) + 5)y \Rightarrow y' = (5\ln(3x) + 5)(3x)^{5x}$.

   (r) Use logarithmic differentiation $y = x^{\ln x} \Rightarrow \ln y = \ln x^{\ln x} = \ln x\ln x = (\ln x)^2 \Rightarrow \frac{1}{y}y' = 2\ln x\frac{1}{x} \Rightarrow y' = (2\ln x\frac{1}{x})x^{\ln x}$.

   (s) Use logarithmic differentiation $\ln y = \ln(\ln x)^x = x\ln(\ln x) \Rightarrow \frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x}\frac{1}{x}x \Rightarrow y' = \left(\ln(\ln x) + \frac{1}{\ln x}\right)y \Rightarrow y' = (\ln(\ln x) + \frac{1}{\ln x})(\ln x)^x$.

   (t) Use logarithmic differentiation $\ln y = \ln(3x+2)^{2x-1} = (2x-1)\ln(3x+1) \Rightarrow \frac{1}{y}y' = 2\ln(3x+1) + \frac{3(2x-1)}{3x+2} \Rightarrow y' = \left(2\ln(3x + 1) + \frac{3(2x-1)}{3x+2}\right)y \Rightarrow y' = (2\ln(3x + 2) + \frac{3(2x-1)}{3x+2})(3x + 2)^{2x-1}$.

2. Solve the equations for x:

 (a) $2^{x-1} = 5$ (b) $3^{2x+3} = 7$

 (c) $e^{3x+4} = 2$ (d) $(3.2)^x = 64.6$

 (e) $\log_3(x+4) = 1$ (f) $\log_5(x^2 + 9) = 2$

 (g) $\ln(\ln x) = 0$ (h) $\ln(x+2) + \ln e^3 = 7$

Derivatives of Exponential Functions and Logarithmic Functions
... Set 3

Answers

2. (a) Take $\log_2$ of both sides. Get $x - 1 = \log_2(5) \Rightarrow x = \log_2(5) + 1 = 3.32$. Alternatively, take ln of both sides and get $(x - 1)\ln 2 = \ln 5 \Rightarrow x = \frac{\ln 5}{\ln 2} + 1 = 3.32$.

(b) Take $\log_3$ of both sides, get $2x + 3 = \log_3(7)$. Solve for x and get $x = \frac{\log_3(7) - 3}{2} = -.61$.

(c) Take ln of both sides. Get $3x + 4 = \ln 2 \Rightarrow x = \frac{\ln 2 - 4}{3} = -1.1$.

(d) $(3.2)^x = 64.6 \Rightarrow x \ln 3.2 = \ln 64.6 \Rightarrow x = \frac{\ln 64.6}{\ln 3.2} = 3.58$.

(e) $\log_3(x + 4) = 1 \Rightarrow 3^{\log_3(x+4)} = 3^1 \Rightarrow x + 4 = 3 \Rightarrow x = -1$.

(f) $\log_5(x^2 + 9) = 2 \Rightarrow 5^{\log_5(x^2+9)} = 5^2 \Rightarrow x^2 + 9 = 25 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$.

(g) $\ln(\ln x) = 0 \Rightarrow \ln x = e^0 \Rightarrow \ln x = 1 \Rightarrow x = e^1 = e \approx 2.72$

(h) Note that $\ln e^3$ simplifies as 3. Thus $\ln(x + 2) + 3 = 7 \Rightarrow \ln(x + 2) = 4 \Rightarrow x + 2 = e^4 \Rightarrow x = e^4 - 2 = 52.6$.

3. This problem deals with functions called the hyperbolic sine and the hyperbolic cosine. These functions occur in the solutions of some differential equations that appear in electromagnetic theory, heat transfer, fluid dynamics, and special relativity. Hyperbolic sine and cosine are defined as follows.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Find derivatives of $\sinh x$ and $\cosh x$ and express your answers in terms of $\sinh x$ and $\cosh x$. Use those formulas to find derivatives of $y = x \sinh x$ and $y = \cosh(x^2)$.

Answers

3. The derivative of $\sinh x = \frac{1}{2}(e^x - e^{-x})$ is $\frac{1}{2}(e^x - e^{-x}(-1)) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$. Similarly, obtain that the derivative of $\cosh x$ is $\sinh x$. Using the product rule obtain that the derivative of $y = x \sinh x$ is $y' = \sinh x + x \cosh x$. Using the chain rule obtain that the derivative of $y = \cosh(x^2)$ is $y' = \sinh(x^2)(2x) = 2x \sinh x$.