

The 2nd Derivative Test

... Set 2

Second-derivative tests

OVERVIEW: In this section we use second derivatives to determine the open intervals on which graphs of functions are concave up and on which they are concave down, to find inflection points of curves, and to test for local maxima and minima at critical points.

Topics:

- The Second-Derivative Test for concavity
- Inflection points
- The Second-Derivative Test for local maxima and minima

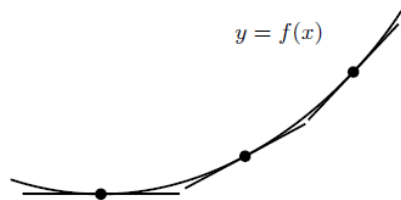
The Second-Derivative Test for concavity

The graph in Figure 1 is said to be CONCAVE UP and the graph in Figure 2 is said to be CONCAVE DOWN, according to the following definition.

Definition 1 (Concavity)

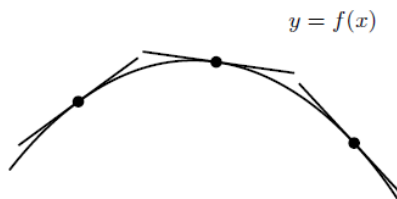
(a) If the slope $f'(x)$ of the tangent line at x to the graph of $y = f(x)$ increases as x increases across an open interval, then graph of the function is concave up on the interval.

(b) If the slope $f'(x)$ of the tangent line at x to the graph of $y = f(x)$ decreases as x increases across an open interval, then the graph of the function is concave down on the interval.



Slope increasing
Graph concave up

FIGURE 1



Slope decreasing
Graph concave down

FIGURE 2

We can find the intervals in which the graph of a function is concave up and the intervals where it is concave down by studying the function's second derivative:

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Theorem 1 (The Second-Derivative Test for concavity)

(a) If $f''(x)$ exists and is positive on an open interval, then the graph of $y = f(x)$ is concave up on the interval.

(b) If $f''(x)$ exists and is negative on an open interval, then the graph of $y = f(x)$ is concave down on the interval.

Proof: If $f''(x)$ is positive on an open interval, then since $f''(x)$ is the derivative of $f'(x)$, Theorem 3 of Section 3.1, applied to $y = f'(x)$, shows that $f'(x)$ is increasing on the interval, so that the graph $y = f(x)$ is concave up on the interval by Definition 1. Similarly, if $f''(x)$ is negative on an open interval, then $f'(x)$ is decreasing and the graph $y = f(x)$ is concave down on the interval. QED

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Example 1 Find the largest open intervals on which the graph of $f(x) = x^3 - 3x^2 + 4$ is concave up and on which it is concave down.

SOLUTION The first and second derivatives of $f(x) = x^3 - 3x^2 + 4$ are

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 4) = 3x^2 - 6x$$
$$f''(x) = \frac{d}{dx}(3x^2 - 6x) = 6x - 6 = 6(x - 1). \quad (1)$$

The second derivative (1) is zero at $x = 1$, negative for $x < 1$, and positive for $x > 1$ (Figure 3). By Theorem 1 above, the graph of $y = f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$. This can be seen from its graph in Figure 4. \square

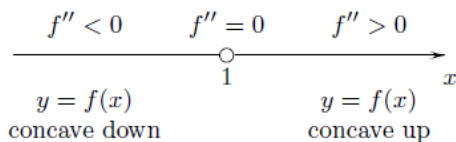


FIGURE 3

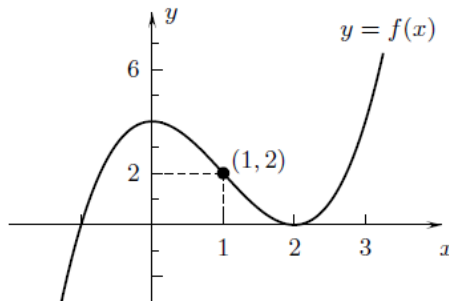


FIGURE 4

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Inflection points

The point $(1, 2)$ at $t = 1$ on the graph of $f(x) = x^3 - 3x^2 + 4$ in Figure 4 where the curve switches from concave up to concave down is called an INFLECTION POINT of the graph, according to the following definition:

Definition 2 (Inflection points) An inflection point on the graph of $y = f(x)$ is a point $(x_0, f(x_0))$ where the graph has a tangent line and is such that either the graph is concave up on an open interval (a, x_0) to the left of x_0 and concave down on an open interval (x_0, b) to the right of x_0 or the graph is concave down on an open interval to the left of x_0 and concave up on an open interval to the right of x_0 .

Example 2 Find the inflection point of $y = 3 + x^2 - 8/x$.

SOLUTION The first derivative of $y = 3 + x^2 - 8x^{-1}$ is $y' = \frac{d}{dx}(x^2 - 8x^{-1}) = 2x + 8x^{-2}$, and its second derivative is

$$y'' = \frac{d}{dx}(2x + 8x^{-2}) = 2 - 16x^{-3} = 2 - \frac{16}{x^3} = \frac{2(x^3 - 8)}{x^3}. \quad (2)$$

The second derivative (1) can change sign only at $x = 0$ where its denominator is zero or at $x = 2$ where its numerator is zero. These values set off three open intervals on which the sign of $f''(x)$ is constant. We could determine the signs by studying the signs of the numerator and denominator of (2) or by calculating sample values. We will use the latter approach and calculate $f''(x)$ at $x = -1$ in $(-\infty, 0)$, at $x = 1$ in $(0, 2)$, and at $x = 3$ in $(2, \infty)$:

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$$y''(-1) = \frac{2[(-1)^3 - 8]}{(-1)^3} = 18 > 0 \implies y''(x) > 0 \text{ for } x < 0$$

$$y''(1) = \frac{2[1^3 - 8]}{1^3} = -14 < 0 \implies y''(x) < 0 \text{ for } 0 < x < 2$$

$$y''(3) = \frac{2[3^3 - 8]}{3^3} = \frac{38}{27} > 0 \implies y''(x) > 0 \text{ for } x > 2.$$

This information is shown above the x -axis in Figure 5. By Theorem 1, the graph is concave up on $(-\infty, 0)$, concave down on $(0, 2)$, and concave up on $(2, \infty)$. The one inflection point is $(2, 3)$ where $x = 2$ and the value of the function is $y(2) = 3 + 2^2 - 8/2 = 3$ (Figure 6). There is no inflection point at $x = 0$ because the function is not defined there. \square

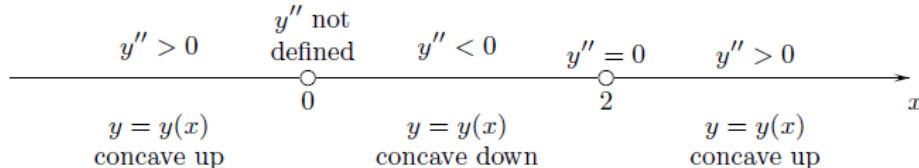
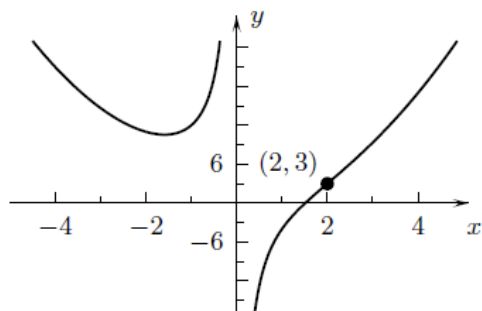
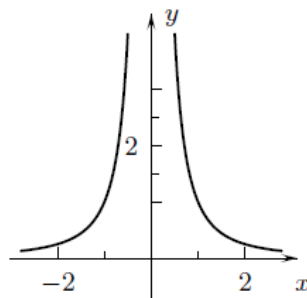


FIGURE 5



$$y = 3 + x^2 - 8/x$$

FIGURE 6



$$y = 1/x^2$$

FIGURE 7

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USING THE FIRST DERIVATIVE: The first derivative is $g'(x) = \frac{d}{dx}(4x^3 - x^4) = 12x^2 - 4x^3 = 4x^2(3 - x)$. Since it is continuous for all x , it has constant sign on each of the open intervals $(-\infty, 0)$, $(0, 3)$, and $(3, \infty)$ set off by the points $x = 0$ and $x = 3$ where it is zero. We determine the signs by finding the signs of the factors $4x^2$ and $3 - x$:

	$4x^2$	$3 - x$	$g'(x) = 4x^2(3 - x)$
$x < 0$	> 0	> 0	> 0
$0 < x < 3$	> 0	> 0	> 0
$x > 3$	> 0	< 0	< 0

By Theorem 3 of Section 3.1, g is increasing on $(-\infty, 0)$ and on $(0, 3)$, and is decreasing on $(3, \infty)$ (Figure 8). Because g is continuous on $(-\infty, \infty)$, it has a global maximum of $g(3) = 3(3^3) - 3^4 = 27$ at $x = 3$.

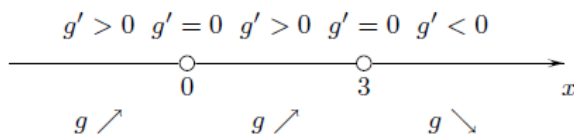


FIGURE 8

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USING THE SECOND DERIVATIVE: The second derivative $g''(x) = \frac{d}{dx}(12x^2 - 4x^3)$
 $= 24x - 12x^2 = 12x(2 - x)$ has constant sign on each of the open intervals $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ set off by its zeros. We could determine the signs of $g''(x)$ by studying the signs of its factors, as we did for $g'(x)$ above. Instead, we calculate a sample value in each interval:

$$g''(-1) = 12(-1)(2 + 1) = -36 < 0 \quad \Rightarrow \quad g''(x) < 0 \text{ in } (-\infty, 0)$$

$$g''(1) = 12(1)(2 - 1) = 12 > 0 \quad \Rightarrow \quad g''(x) > 0 \text{ in } (0, 2)$$

$$g''(3) = 12(3)(2 - 1) = 36 > 0 \quad \Rightarrow \quad g''(x) < 0 \text{ in } (2, \infty).$$

The second derivative $g''(x)$ is negative and the graph is concave down on $(-\infty, 0)$; $g''(x)$ is positive and the graph is concave up on $(0, 2)$; and $g''(x)$ is negative and the graph is concave down on $(2, \infty)$ (Figure 9). The graph has inflection points at $x = 0$ and $x = 2$. They are $(0, 0)$ and $(2, 16)$ since $g(0) = 4(0^3) - 0^4 = 0$ and $g(2) = 4(2^3) - 2^4 = 16$. To draw the graph in Figure 10, we use the information obtained above and plot the inflection points and the point $(3, 27)$ at the global maximum. \square

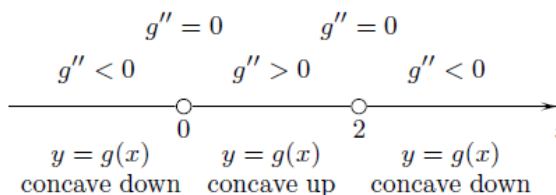


FIGURE 9

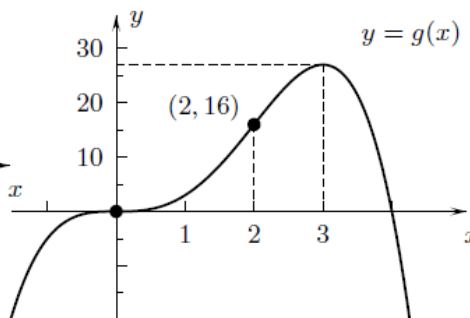


FIGURE 10

- ^c **Question 2** Illustrate that the graph of the polynomial $y = 4x^3 - x^4$ from Example 3 looks the graph $y = 4x^3$ of its term of lowest degree near $x = 0$ by generating the curves together in the window $-1.5 \leq x \leq 5, -20 \leq y \leq 35$.

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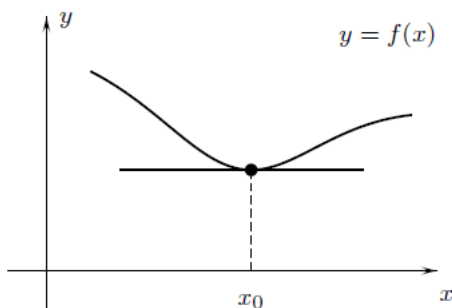
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The Second-Derivative Test at a local maximum or minimum

We usually show that a function $y = f(x)$ has a local maximum or minimum at a critical point x_0 by determining whether the derivative is positive or negative in open intervals to the left and to the right of x_0 . In some cases, however, it is more convenient to calculate the second derivative at x_0 and use the following result.

Theorem 2 (The Second-Derivative Test for a local maximum or minimum)

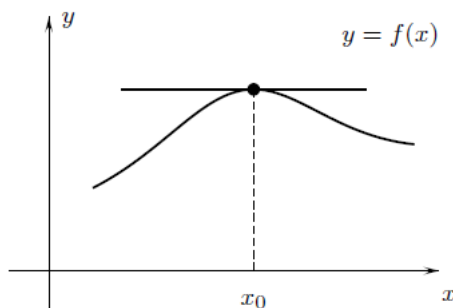
- (a) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $y = f(x)$ has a local minimum at x_0 (Figure 11).
 - (b) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $y = f(x)$ has a local maximum at x_0 (Figure 12).
-



$$f'(x_0) = 0 \text{ and } f''(x_0) > 0$$

Local minimum

FIGURE 11



$$f'(x_0) = 0 \text{ and } f''(x_0) < 0$$

Local maximum

FIGURE 12

Proof: To clarify the ideas, we make the additional assumption that $f''(x)$ is either positive or negative on an open interval (a, b) containing x_0 .[†] Then $y = f'(x)$ and $y = f(x)$ are continuous on (a, b) by Theorem 1 of Section 2.5.

[†]Part (a) of Theorem 2, as stated, can be established with the following argument: Because $f'(x_0)$ is zero, $f''(x_0)$ is the limit of $\frac{f'(x) - f'(x_0)}{x - x_0} = \frac{f'(x)}{x - x_0}$ as $x \rightarrow x_0$. If $f''(x_0)$ is positive, then $\frac{f'(x)}{x - x_0}$ is defined and positive on an open interval (a, b) containing x_0 . This implies that $y = f(x)$ is continuous on (a, b) , and that $f'(x)$ is negative on (a, x_0) and positive on (x_0, b) . Thus, $y = f(x)$ is decreasing on $(a, x_0]$ and increasing on $[x_0, b)$ and has a local minimum at x_0 . Part (b) can be established similarly.

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If $f''(x)$ is positive and the graph is concave up on (a, b) , as in Figure 11, then $y = f'(x)$ is increasing on (a, b) by Theorem 2 of Section 5.1. Consequently, because $f'(x_0) = 0$, the first derivative $f'(x)$ is negative on (a, x_0) and positive on (x_0, b) , so that by Theorem 3 of Section 3.1, $y = f(x)$ is decreasing on $(a, x_0]$ and increasing on $[x_0, b)$ and has a local minimum at x_0 .

If $f''(x)$ is negative and the graph is concave down on (a, b) , as in Figure 12, then $f'(x)$ is increasing on (a, b) and zero at x_0 , so that $f'(x)$ is positive on (a, x_0) and negative on (x_0, b) . This implies that $y = f(x)$ is increasing on $(a, x_0]$ and decreasing on $[x_0, b)$ and has a local maximum at x_0 . QED

Example 4 (a) Show that $h(x) = 3 - 2x + \frac{1}{2}x^2 + \frac{1}{4}x^4$ has a critical point at $x = 1$.
 (b) Does h have a local maximum or a local minimum at $x = 1$?

SOLUTION (a) The first derivative $h'(x) = \frac{d}{dx}(3 - 2x + \frac{1}{2}x^2 + \frac{1}{4}x^4) = -2 + x + x^3$ has the value $h'(1) = -2 + 1 + 1^3 = 0$, so $x = 1$ is a critical point.

(b) The second derivative $h''(x) = \frac{d}{dx}(-2 + x + x^3) = 1 + 3x^2$ has the value $h''(1) = 1 + 3(1)^2 = 4$, which is positive. By Theorem 2a, $y = h(x)$ has a local minimum at $x = 1$. \square

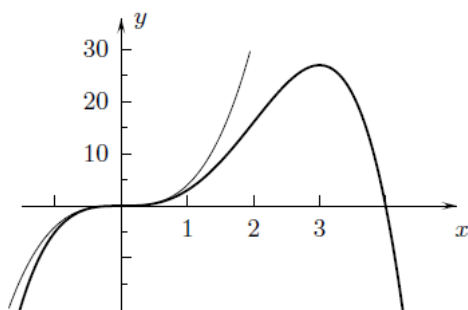
Question 3 Illustrate the fact that $h(x) = 3 - 2x + \frac{1}{2}x^2 + \frac{1}{4}x^4$ of Example 4 has a local minimum at $x = 1$ by generating its graph in the window $-1.5 \leq x \leq 2.5, -1 \leq y \leq 6$.

Responses 3.3

Response 1 The curve $y = 1/x^2$ in Figure 7 is not concave up on $(-\infty, \infty)$ because the slope $-1/x^3$ of its tangent line at x does not increase across the entire interval. The slope is positive for $x < 0$, not defined at $x = 0$, and negative for $x > 0$.

Response 2 Figure R2.

Response 3 Figure R3



$$y = 3 - 2x + \frac{1}{2}x^2 + \frac{1}{4}x^4$$

FIGURE R2

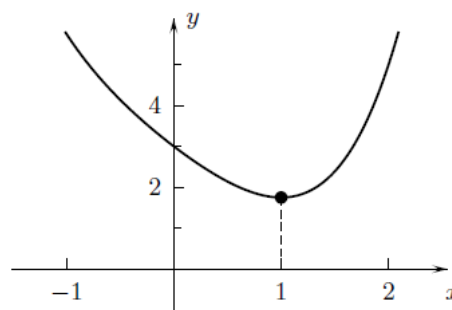


FIGURE R3