# Differential Calculus: The Derivative and Rules of Differentiation

## Limits

**Question 1:** Find $\lim_{x \to 3} f(x)$:

$$f(x) = \frac{x^2 - 9}{x - 3}$$

(A) $+\infty$

(B) -6

(C) 6

(D) Does not exist!

(E) None of the above

**Answer:** (C) Note the the function $f(x) = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x + 3$ is actually a line. However it is important to note the this function is *undefined* at $x = 3$. Why? $x = 3$ requires dividing by zero (which is inadmissible). As $x$ approaches 3 from below and from above, the value of the function $f(x)$ approaches $f(3) = 6$. Thus the limit $\lim_{x \to 3} f(x) = 6$.

**Question 2:** Find $\lim_{x \to 2} f(x)$:

$$f(x) = 1776$$

(A) $+\infty$

(B) 1770

(C) $-\infty$

(D) Does not exist!

(E) None of the above

**Answer:** (E) The limit of any constant function at any point, say $f(x) = C$, where $C$ is an arbitrary constant, is simply $C$. Thus the correct answer is $\lim_{x \to 2} f(x) = 1776$.

**Question 3:** Find $\lim_{x \to 4} f(x)$:

$$f(x) = ax^2 + bx + c$$

(A) $+\infty$

(B) 16a + 4b + c

(C) $-\infty$

(D) Does not exist!

(E) None of the above

**Answer:** (B) Applying the rules of limits:

$$\begin{aligned} \lim_{x\to4} ax^2 + bx + c &= \lim_{x\to4} ax^2 + \lim_{x\to4} bx + \lim_{x\to4} c \\ &= a\left[\lim_{x\to4} x\right]^2 + b\lim_{x\to4} x + c \\ &= 16a + 4b + c \end{aligned}$$

**Answer:** Applying the rules of limits:

**Question 4:** Find $\lim_{x\to8} f(x)$:

$$f(x) = \frac{x^2 + 7x - 120}{x - 7}$$

**Answer:** Applying the rules of limits:

$$\begin{aligned} \lim_{x\to8} \frac{x^2 + 7x - 120}{x - 7} &= \frac{8^2 + 7 * 8 - 120}{8 - 7} \\ &= \frac{120 - 120}{1} & = \frac{0}{1} \\ &= 0 \end{aligned}$$

**Question 5:** Find $\lim_{x\to2} f(x)$:

$$f(x) = \frac{3x^2 - 4x + 6}{x^2 + 8x - 15}$$

**Answer:** Applying the rules of limits:

$$\begin{aligned} \lim_{x\to2} \frac{3x^2 - 4x + 6}{x^2 + 8x - 15} &= \frac{3(2)^2 - 4(2) + 6}{(2)^2 + 8(2) - 15} \\ &= \frac{12 - 8 + 6}{4 + 1} & = \frac{10}{5} \\ &= 2 \end{aligned}$$

**Question 6:** Find $\lim_{x\to\infty} f(x)$:

$$f(x) = \frac{9}{4x^2 - 7}$$

**Answer:** Applying the rules of limits:

$$\begin{aligned} \lim_{x\to\infty} \frac{9}{4x^2 - 7} &= \frac{9}{4(\infty)^2 - 7} \\ &= \frac{9}{4\infty - 7} & = \frac{9}{\infty - 7} = \frac{9}{\infty} \\ &= 0 \end{aligned}$$

## Continuity and Differentiability

**Question 7:** Which of the following functions are *NOT* everywhere continuous:

(A) $f(x) = \frac{x^2-4}{x+2}$

(B) $f(x) = (x+3)^4$

(C) $f(x) = 1066$

(D) $f(x) = mx + b$

(E) None of the above

**Answer:** (A) Remember that, informally at least, a *continuous* function is one in which there are no breaks its curve. A continuous function can be drawn without lifting your pencil from the paper. More formally, a function $f(x)$ is *continuous* at the point $x = a$ if and only if:

1. $f(x)$ is defined at the point $x = a$,
2. the limit $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$

The function $f(x) = \frac{x^2-4}{x+2}$ is not everywhere continuous because the function is not defined at the point $x = -2$. It is worth noting that $\lim_{x \to -2} f(x)$ does in fact exist! **The existence of a limit at a point does not guarantee that the function is continuous at that point!**

**Question 8:** Which of the following functions are continuous:

(A) $f(x) = |x|$

(B) $f(x) = \begin{cases} 3 & x < 4 \\ \frac{1}{2}x + 3 & x \geq 4 \end{cases}$

(C) $f(x) = \frac{1}{x}$

(D) $f(x) = \begin{cases} \ln x & x < 0 \\ 0 & x = 0 \end{cases}$

(E) None of the above

**Answer:** (A) The absolute value function $f(x) = |x|$ is defined as:

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Does this function satisfy the requirements for continuity? Yes! The critical point to check is $x = 0$. Note that the function is defined at $x = 0$; the $\lim_{x \to 0} f(x)$ exists; and that $\lim_{x \to 0} f(x) = 0 = f(0)$.

**Question 9:** Which of the following functions are *NOT* differentiable:

(A) $f(x) = |x|$

(B) $f(x) = (x + 3)^4$

(C) $f(x) = 1066$

(D) $f(x) = mx + b$

(E) None of the above

**Answer:** (A) Remember that continuity is a *necessary* condition for differentiability (i.e., every differentiable function is continuous), but continuity is not a *sufficient* condition to ensure differentiability (i.e., not every continuous function is differentiable). Case in point is $f(x) = |x|$. This function is in fact continuous (see previous question). It is not however differentiable at the point $x = 0$. Why? The point $x = 0$ is a cusp (or kink). There are an infinite number of lines that could be tangent to the function $f(x) = |x|$ at the point $x = 0$, and thus the derivative of $f(x)$ would have an infinite number of possible values.