Combining Functions

In this section, we will look at the combination of functions as well as the composition of functions. In other words, we will be taking two functions and combining them to form one new function.

Combining Functions Using Algebra:

Using algebraic operations to combine functions is really quite simple. We'll begin with a formal definition, then go on to some examples.

Def: Algebra of Functions:

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined in the following way:

$(f+g)(x) = f(x)+g(x)$ $\quad D: A \cap B$

$(f-g)(x) = f(x)-g(x)$ $\quad D: A \cap B$

$(fg)(x) = f(x)g(x)$ $\quad D: A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\quad D: \{x \in A \cap B \mid g(x) \neq 0\}$

Recall: The symbol $\cap$ means "intersection" and includes all values, which are in *both* domains A and B. It is also important to note that with division, the function in the denominator cannot have a value of zero, because this would require division by zero! Therefore, the domain must exclude this value.

Example 1: Find $f + g$, $f - g$, fg, and f/g and state their domains.

$$f(x) = x^2$$

$$g(x) = \frac{1}{x} + 3$$

Solution:

(a) $f + g$

$$(f + g)(x) = f(x) + g(x)$$
$$= x^2 + \left(\frac{1}{x} + 3\right)$$
$$= x^2 + \frac{1}{x} + 3$$

The domain of f is the set of all real numbers, and the domain of g is the set of all real numbers excluding $x = 0$. Therefore,

Domain $\{x \mid x \neq 0\}$

(b) $f - g$

$$(f - g)(x) = f(x) - g(x)$$
$$= x^2 - \left(\frac{1}{x} + 3\right)$$
$$= x^2 - \frac{1}{x} - 3$$

Domain $\{x \mid x \neq 0\}$

(c) *fg*

$$(fg)(x) = f(x) \cdot g(x)$$

$$= \left(x^2\right)\left(\frac{1}{x} + 3\right)$$

$$= \frac{x^2}{x} + 3x^2$$

$$= 3x^2 + x$$

Domain $\{x \mid x \neq 0\}$

(d) *f/g*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$= \frac{x^2}{\frac{1}{x} + 3}$$

$$= \frac{x^2}{\frac{1}{x} + \frac{3x}{x}}$$

$$= \frac{x^2}{\frac{1 + 3x}{x}}$$

$$= \frac{x^3}{1 + 3x}$$

Domain $\left\{x \mid x \neq 0, x \neq -\frac{1}{3}\right\}$

(If $x = -\frac{1}{3}$, then the value of *g(x)* would be zero.)

Composition of Functions:

It can be very beneficial sometimes, to combine functions in such a way that the value of the variable in one function is found in terms of a second function (or perhaps the same function again). In other words, you begin with a certain value *t*. Evaluate a function *s* using this *t* value. The result is now used in a second function *r*, and a new value is achieved. A new function can be formed using the combination of the original two functions to make the evaluations simpler.

In function notation, this would look like:

$$r(s(t)) = q(t)$$

where $q(t)$ is the new function formed using the composition of functions.

Def: Composition of Functions

Given two functions *f* and *g*, the **composite function** $f \circ g$ (also called the **composition** of *f* and *g*) is defined by:

$$(f \circ g)(x) = f(g(x))$$

Note: The domain is defined whenever both $g(x)$ and $f(g(x))$ are defined.

I think this will become clearer as we go through some examples.

Example 2: Let $f(x)=\sqrt{x}$ and $g(x)=2x+5$.
Find $f\circ g$, $g\circ f$, $f\circ f$, and $g\circ g$, and state the domain of each.

Solution:

(i)

$$\begin{aligned} f\circ g(x) &= f(g(x)) \\ &= f(2x+5) \\ &= \sqrt{2x+5} \end{aligned}$$

Notice that all we are doing is substituting the actual function *g(x)* (in this case, 2*x* + 5) in for *x* in the function *f*.

We know that *g(x)* is defined for the set of all real numbers; therefore, we only need to consider *f(g(x))* in this case. We mustn't take the square root of a negative number if we want a real number solution. The domain must be the set of all real numbers such that (2*x* + 5) is greater than or equal to zero.

$$\begin{aligned} 2x+5 &\geq 0 \\ 2x &\geq -5 \\ x &\geq -\frac{5}{2} \end{aligned}$$

so $x \geq -2\frac{1}{2}$

The domain of $f\circ g$ is $\left\{x \mid x \geq -2\frac{1}{2}\right\} = [-2.5, \infty)$

(ii)

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= g(\sqrt{x}) \\ &= 2\sqrt{x} + 5 \end{aligned}$$

We know that *f(x)* is defined when $x \geq 0$, and *g(f(x))* is also defined when $x \geq 0$.

The domain of $g \circ f$ is $[0, \infty)$.

(iii)

$$\begin{aligned} f \circ f &= f(f(x)) \\ &= f(\sqrt{x}) \\ &= \sqrt{\sqrt{x}} \end{aligned}$$

The domain of *f(x)* is $[0, \infty)$. This is the only value we need to be concerned with in this case, since *f(x)* will always be a positive number. Therefore, *f(f(x))* will always be defined if *f(x)* is defined.

The domain of $f \circ f$ is $[0, \infty)$.

(iv)

$$\begin{aligned} g \circ g &= g(g(x)) \\ &= g(2x + 5) \\ &= 2(2x + 5) + 5 \\ &= 4x + 10 + 5 \\ &= 4x + 15 \end{aligned}$$

The domain of *g(x)* is the set of all real numbers. Also, *g(g(x))* is defined for the set of all real numbers. Therefore, the domain of $g \circ g$ is the set of all real numbers.

Example 3: Evaluate the expression $(g \circ f)(4)$ if $g(x) = 16x - 3$ and $f(x) = \dfrac{1}{x}$.

Solution: There are a couple of ways to work a problem like this. We can either evaluate *f(x)* at the value $x = 4$, then substitute this value into the function *g(x)*. We can also find the composition function first, then evaluate this function at the value $x = 4$.

Using the second option,

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= g\left(\frac{1}{x}\right) \\ &= 16\left(\frac{1}{x}\right) - 3 \\ &= \frac{16}{x} - 3 \end{aligned}$$

Evaluating this at $x = 4$, we have

$$\begin{aligned} (g \circ f)(4) &= \frac{16}{4} - 3 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$