

Solving Logarithmic Equations

Deciding How to Solve Logarithmic Equation

When asked to solve a logarithmic equation such as $\log_2(5x + 7) = 5$ or $\log_3(7x + 3) = \log_3(5x + 9)$, the first thing we need to decide is how to solve the problem. Some logarithmic problems are solved by simply dropping the logarithms while others are solved by rewriting the logarithmic problem in exponential form. How do we decide what is the correct way to solve a logarithmic problem? The key is to look at the problem and decide if the problem contains only logarithms or if the problem has terms without logarithms.

If we consider the problem $\log_2(5x + 7) = 5$, this problem contains a term, 5, that does not have a logarithm. So, the correct way to solve these types of logarithmic problems is to rewrite the logarithmic problem in exponential form. If we consider the example $\log_3(7x + 3) = \log_3(5x + 9)$, this problem contains only logarithms. So, the correct way to solve these types of logarithmic problems is to simply drop the logarithms.

Properties of Logarithms Revisited

When solving logarithmic equation, we may need to use the properties of logarithms to simplify the problem first. The properties of logarithms are listed below as a reminder.

Properties for Condensing Logarithms

Property 1: $0 = \log_a 1$ – Zero-Exponent Rule

Property 2: $1 = \log_a a$

Property 3: $\log_a x + \log_a y = \log_a(xy)$ – Product Rule

Property 4: $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ – Quotient Rule

Property 5: $y \log_a x = \log_a x^y$ – Power Rule

Solving Logarithmic Equations Containing Only Logarithms

After observing that the logarithmic equation contains only logarithms, what is the next step?

One-To-One Property of Logarithms

If $\log_b M = \log_b N$, then $M = N$.

This statement says that if an equation contains only two logarithms, on opposite sides of the equal sign, with the same base then the problem can be solved by simply dropping the logarithms.

Let's finish solving the problem $\log_3(7x + 3) = \log_3(5x + 9)$.

$$\log_3(7x + 3) = \log_3(5x + 9).$$

Since the problem has only two logarithms on opposite sides of the equal sign, the problem can be solved by dropping the logarithms.

$$7x + 3 = 5x + 9$$

Drop the logarithms.

$$x = 3$$

Finish solving by subtracting $5x$ from each side, subtracting 3 from each side, and finally dividing each side by 2 .

$$x = 3$$

Check the answer; this is an acceptable answer because we get a positive number when it is plugged back in.

Therefore, the solution to the problem $\log_3(7x + 3) = \log_3(5x + 9)$ is $x = 3$.

Here is another example, solve $\log_7(x - 2) + \log_7(x + 3) = \log_7 14$.

$$\log_7((x - 2)(x + 3)) = \log_7 14$$

This problem can be simplified by using Property 3 which changes the addition of logarithms to multiplication.

$$(x - 2)(x + 3) = 14$$

Drop the logarithms.

$$x^2 - x - 6 = 14$$

Simplify the problem by distributing or FOILing and combining like terms.

$$x^2 - x - 20 = 0$$

Solve the problem by subtracting 14 from each to get it equal to zero, and then factoring or using the quadratic formula to find the values of x.

$$(x + 4)(x - 5) = 0$$

$$x = -4 \text{ or } x = 5$$

$$x = 5$$

Check the answers, only one answer is acceptable because the other answer produces a negative number when it is plugged back in.

Therefore, the solution to the problem $\log_7(x - 2) + \log_7(x + 3) = \log_7 14$ is $x = 5$.

Steps for Solving Logarithmic Equations Containing Only Logarithms

Step 1: Determine if the problem contains only logarithms. If so, go to Step 2. If not, stop and use the Steps for Solving Logarithmic Equations Containing Terms without Logarithms.

Step 2: Use the properties of logarithms to simplify the problem if needed. If the problem has more than one logarithm on either side of the equal sign then the problem can be simplified.

Step 3: Rewrite the problem without the logarithms.

Step 4: Simplify the problem.

Step 5: Solve for x .

Step 6: Check your answer(s). Remember we cannot take the logarithm of a negative number, so we need to make sure that when we plug our answer(s) back into the original equation we get a positive number. Otherwise, we must drop that answer(s).

Solving Logarithmic Equations Containing Terms without Logarithms

After observing that the logarithmic equation contains terms without logarithms, what is the next step? The next step is to simplify the problem using the properties of logarithms and then to rewrite the logarithmic problem in exponential form. After rewriting the problem in exponential form we will be able to solve the resulting problem.

Let's finish solving the problem $\log_2(5x + 7) = 5$ from earlier.

$$\log_2(5x + 7) = 5$$

This problem does not need to be simplified because there is only one logarithm in the problem.

$$5x + 7 = 2^5$$

Rewrite the problem in exponential form by moving the base of the logarithm to the other side.

$$5x + 7 = 32$$

Simplify the problem by raising 2 to the fifth power.

$$x = 5$$

Solve for x by subtracting 7 from each side and then dividing each side by 5.

$$x = 5$$

Check the answer; this is an acceptable answer because we get a positive number when it is plugged back in.

Therefore, the solution to the problem $\log_2(5x + 7) = 5$ is $x = 5$.

Here is another example, solve $\ln(4x - 1) = 3$.

$$\ln(4x - 1) = 3$$

This problem does not need to be simplified because there is only one logarithm in the problem.

$$4x - 1 = e^3$$

Rewrite the problem in exponential form by moving the base of the logarithm to the other side. For natural logarithms the base is e.

$$4x - 1 \approx 20.085537$$

Simplify the problem by cubing e. Round the answer as appropriate, these answers will use 6 decimal places.

$$x \approx 5.271384$$

Solve for x by adding 1 to each side and then dividing each side by 4.

$$x \approx 5.271384$$

Check the answer; this is an acceptable answer because we get a positive number when it is plugged back in.

Therefore, the solution to the problem $\ln(4x - 1) = 3$ is $x \approx 5.271384$.

Steps for Solving Logarithmic Equations Containing Terms without Logarithms

Step 1: Determine if the problem contains only logarithms. If so, stop and use Steps for Solving Logarithmic Equations Containing Only Logarithms. If not, go to Step 2.

Step 2: Use the properties of logarithms to simplify the problem if needed. If the problem has more than one logarithm on either side of the equal sign then the problem can be simplified.

Step 3: Rewrite the problem in exponential form.

Step 4: Simplify the problem.

Step 5: Solve for x.

Step 6: Check your answer(s). Remember we cannot take the logarithm of a negative number, so we need to make sure that when we plug our answer(s) back into the original equation we get a positive number. Otherwise, we must drop that answer(s).

Example 1: Solve $\log_3(9x + 2) = 4$

$$\log_3(9x + 2) = 4$$

This problem contains terms without logarithms.

$$\log_3(9x + 2) = 4$$

This problem does not need to be simplified because there is only one logarithm in the problem.

$$9x + 2 = 3^4$$

Rewrite the problem in exponential form by moving the base of the logarithm to the other side.

$$9x + 2 = 81$$

Simplify the problem by raising 3 to the fourth power.

$$x = \frac{79}{9}$$

Solve for x by subtracting 2 from each side and then dividing each side by 9.

$$x = \frac{79}{9}$$

Check the answer; this is an acceptable answer because we get a positive number when it is plugged back in.

Therefore, the solution to the problem $\log_3(9x + 2) = 4$ is $x = \frac{79}{9}$.

Example 2: Solve $\log_4 x + \log_4(x-12) = 3$

$$\log_4 x + \log_4(x-12) = 3$$

This problem contains terms without logarithms.

$$\log_4(x(x-12)) = 3$$

This problem can be simplified by using Property 3 which changes the addition of logarithms to multiplication.

$$x(x-12) = 4^3$$

Rewrite the problem in exponential form by moving the base of the logarithm to the other side.

$$x^2 - 12x = 64$$

Simplify the problem by distributing and cubing the 4.

$$x^2 - 12x - 64 = 0$$

Solve the problem by subtracting 64 from each to get it equal to zero, and then factoring or using the quadratic formula to find the values of x.

$$(x+4)(x-16) = 0$$

$$x = -4 \text{ or } x = 16$$

$$x = 16$$

Check the answers, only one answer is acceptable because the other answer produces a negative number when it is plugged back in.

Therefore, the solution to the problem $\log_4 x + \log_4(x-12) = 3$ is $x = 16$.

Example 3: Solve $\log_4(2x+1) = \log_4(x+2) - \log_4 3$

$$\log_4(2x+1) = \log_4(x+2) - \log_4 3$$

This problem contains only logarithms.

$$\log_4(2x+1) = \log_4\left(\frac{x+2}{3}\right)$$

This problem can be simplified by using Property 4 which changes the subtraction of logarithms to division.

$$2x+1 = \frac{x+2}{3}$$

Drop the logarithms.

$$3(2x+1) = x+2$$

Simplify the problem by cross-multiplying to get rid of the fractions.

$$6x+3 = x+2$$

Solve the problem by distributing the 3, subtracting x from each side, subtracting 3 from each side, and finally dividing by 5.

$$x = -\frac{1}{5}$$

No Solution

Check the answers, this problem has "No Solution" because the only answer produces a negative number and we can't take the logarithm of a negative number.

Therefore, the problem $\log_4(2x+1) = \log_4(x+2) - \log_4 3$ has no solution.

Example 4: Solve $\log(5x - 11) = 2$

$$\log(5x - 11) = 2$$

This problem contains terms without logarithms.

$$\log(5x - 11) = 2$$

This problem does not need to be simplified because there is only one logarithm in the problem.

$$5x - 2 = 10^2$$

Rewrite the problem in exponential form by moving the base of the logarithm to the other side. For common logarithms the base is 10.

$$5x - 2 = 100$$

Simplify the problem by squaring the 10.

$$x = \frac{102}{5}$$

Solve for x by adding 2 to each side and then dividing each side by 5.

$$x = \frac{102}{5}$$

Check the answer; this is an acceptable answer because we get a positive number when it is plugged back in.

Therefore, the solution to the problem $\log(5x - 11) = 2$ is $x = \frac{102}{5}$.

Example 5: Solve $\log_2(x+1) - \log_2(x-4) = 3$

$$\log_2(x+1) - \log_2(x-4) = 3$$

This problem contains terms without logarithms.

$$\log_2\left(\frac{x+1}{x-4}\right) = 3$$

This problem can be simplified by using Property 4 which changes the subtraction of logarithms to division.

$$\frac{x+1}{x-4} = 2^3$$

Rewrite the problem in exponential form by moving the base of the logarithm to the other side.

$$\frac{x+1}{x-4} = 8$$

Simplify the problem by cubing the 2.

$$x+1 = 8(x-4)$$

Solve for x by cross-multiplying, distributing, subtracting $8x$ from each side, subtracting 1 from each side, and finally dividing each side by -7 .

$$x+1 = 8x-32$$

$$x = \frac{33}{7}$$

$$x = \frac{33}{7}$$

Check the answer; this is an acceptable answer because we get a positive number when it is plugged back in.

Therefore, the solution to the problem $\log_2(x+1) - \log_2(x-4) = 3$ is $x = \frac{33}{7}$.

Example 6: Solve $\log_6(x + 4) + \log_6(x - 2) = \log_6(4x)$

$$\log_6(x + 4) + \log_6(x - 2) = \log_6(4x)$$

This problem contains only logarithms.

$$\log_6((x + 4)(x - 2)) = \log_6(4x)$$

This problem can be simplified by using Property 3 which changes the addition of logarithms to multiplication.

$$(x + 4)(x - 2) = 4x$$

Drop the logarithms.

$$x^2 + 2x - 8 = 4x$$

Simplify the problem by distributing or FOILING and combining like terms.

$$x^2 - 2x - 8 = 0$$

Solve the problem by subtracting $4x$ from each to get it equal to zero, and then factoring or using the quadratic formula to find the values of x .

$$(x + 2)(x - 4) = 0$$

$$x = -2 \text{ or } x = 4$$

$$x = 4$$

Check the answers, only one answer is acceptable because the other answer produces a negative number when it is plugged back in.

Therefore, the solution to the problem $\log_6(x + 4) + \log_6(x - 2) = \log_6(4x)$ is $x = 4$.

Practice Problems

Now it is your turn to try a few practice problems on your own. Work on each of the problems below and then click on the link at the end to check your answers.

Problem 1: Solve: $\log_5(4x + 11) = 2$

Problem 2: Solve: $\log_2(x + 5) - \log_2(2x - 1) = 5$

Problem 3: Solve: $\log_8 x + \log_8(x + 6) = \log_8(5x + 12)$

Problem 4: Solve: $\log_6 x + \log_6(x - 9) = 2$

Problem 5: Solve: $\ln(6x - 5) = 3$

Problem 6: Solve: $\log_4(3x - 2) - \log_4(4x + 1) = 2$

Problem 7: Solve: $\log_3(x^2 - 6x) = 3$

Problem 8: Solve: $\log(x - 2) - \log(2x - 3) = \log 2$

[Solutions to Practice Problems](#)