SAT: Numbers and Operations

Numbers and Operations

1. A bag contains tomatoes that are either green or red. The ratio of green tomatoes to red tomatoes in the bag is 4 to 3. When five green tomatoes and five red tomatoes are removed, the ratio becomes 3 to 2. How many red tomatoes were originally in the bag?

(A) 12
(B) 15
(C) 18
(D) 24
(E) 30

2. If each digit in an integer is greater than the digit to the left, the integer is said to be "monotonic". For example, 12 is a monotonic integer since $2 > 1$. How many positive two-digit monotonic integers are there?

(A) 28
(B) 32
(C) 36
(D) 40
(E) 44

$$a,\ 2a-1,\ 3a-2,\ 4a-3,\ \ldots$$

3. For a particular number a, the first term in the sequence above is equal to a, and each term thereafter is 7 greater than the previous term. What is the value of the 16^{th} term in the sequence?

4. If p is a prime number, how many factors does p^3 have?

(A) One
(B) Two
(C) Three
(D) Four
(E) Five

5. How many integers between 10 and 500 begin and end in 3?

6. A particular integer N is divisible by two different prime numbers p and q. Which of the following must be true?

I. N is not a prime number.

II. N is divisible by pq.

III. N is an odd integer.

(A) I only
(B) II only
(C) I and II only
(D) I and III only
(E) I, II, and III

Answers

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1. B (Estimated Difficulty Level: 5)

The number of green and red tomatoes are $4n$ and $3n$, respectively, for some integer n. In this way, we can be sure that the green-to-red ratio is $4n/3n = 4/3$. We need to solve the equation:

$$\frac{4n-5}{3n-5} = \frac{3}{2}.$$

Cross-multiplying, $8n - 10 = 9n - 15$ so that $n = 5$. There were $3n$, or 15, red tomatoes in the bag.

Working with the answers may be easier. If answer A is correct, then there were 16 green tomatoes and 12 red tomatoes, in order to have the 4 to 3 ratio. But removing five of each gives 11 green and 7 red, which is not in the ratio of 3 to 2. If answer B is correct, then there were 20 green tomatoes and 15 red tomatoes, since $20/15 = 4/3$. Removing five of each gives 15 green and 10 red, and $15/10 = 3/2$, so answer B is correct.

2. C (Estimated Difficulty Level: 4)

From 10 to 19, 12 and up (eight numbers) are monotonic. Among the numbers from 20 to 29, seven (23 and up) are monotonic. If you can see a pattern in counting problems like this, you can save a lot of time. Here, the 30s will have 6 monotonic numbers, the 40s will have 5, and so forth. You should find $8+7+6+5+4+3+2+1+0 = 36$ total monotonic numbers.

3. 113 (Estimated Difficulty Level: 5)

Since the second term is 7 greater than the first term, $(2a-1) - a = 7$ so that $a = 8$. The sequence is 8, 15, 22, ... You can either continue to write out the sequence until the 16^{th} term, or realize that the 16^{th} term is $16a - 15 = 16(8) - 15 = 128 - 15 = 113$.

4. D (Estimated Difficulty Level: 4)

The answer must be true for any value of p, so plug in an easy (prime) number for p, such as 2. The factors of $2^3 = 8$ are 1, 2, 4, and 8, so answer D is correct.

In general, since p is prime, the only numbers that go into p^3 without a remainder are 1, p, p^2, and p^3.

5. 11 (Estimated Difficulty Level: 4)

For the two-digit numbers, only 33 begins and ends in 3. For three-digit numbers, the only possibilities are: 303, 313, ..., 383, and 393. We found ten three-digit numbers, and one two-digit number, for a total of 11 numbers that begin and end in 3.

Yes, this was a counting problem soon after another counting problem. But this one wasn't so bad, was it?

6. C (Estimated Difficulty Level: 4)

This type of SAT math question contains three separate mini-problems. (This kind of question is also known as "one of those annoying, long, SAT math questions with roman numerals"). Let's do each mini-problem in order.

First, recall that a prime number is only divisible by itself and 1, and that 1 is not a prime number. So, statement I must be true, since a number that can be divided by two prime numbers can't itself be prime.

Next, recall that every number can be written as a product of a particular bunch of prime numbers. Let's say that N is divisible by 3 and 5. Then, N is equal to $3 \cdot 5 \cdot p_1 \cdot p_2 \cdots$, where p_1, p_2, etc. are some other primes. So, N is divisible by $3 \cdot 5 = 15$. Statement II must be true.

Finally, remember that 2 is a prime number. So, N could be 6, since $6 = 2 \cdot 3$. Statement III isn't always true, making C the correct answer.

Numbers and Operations

7. A perfect square is an integer that is the square of an integer. Suppose that m and n are positive integers such that $mn > 15$. If $15mn$ is a perfect square, what is the least possible value of mn?

8. M is a set of six consecutive even integers. When the least three integers of set M are summed, the result is x. When the greatest three integers of set M are summed, the result is y. Which of the following is true?

(A) $y = x - 18$
(B) $y = x + 18$
(C) $y = 2x$
(D) $y = 2x + 4$
(E) $y = 2x + 6$

9. A three-digit number, XYZ, is formed of three different non-zero digits X, Y, and Z. A new number is formed by rearranging the same three digits. What is the greatest possible difference between the two numbers? (For example, 345 could be rearranged into 435, for a difference of $435 - 345 = 90$.)

10. An integer is subtracted from its square. The result could be which of the following?

(A) A negative integer.
(B) An odd integer.
(C) The product of two consecutive even integers.
(D) The product of two consecutive odd integers.
(E) The product of two consecutive integers.

Answers

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7. 60 (Estimated Difficulty Level: 5)

First, note that $15mn = 3 \cdot 5 \cdot mn$. We need $\sqrt{3 \cdot 5 \cdot mn}$ to be an integer. We could have, for example, $m = 3$ and $n = 5$ since $\sqrt{3 \cdot 5 \cdot 3 \cdot 5} = \sqrt{3^2 \cdot 5^2} = 15$, except that the problem requires that $mn > 15$. (This is a hard problem for a reason, after all!) If $m = 3 \cdot 2$ and $n = 5 \cdot 2$ then $15mn = (3 \cdot 5)(3 \cdot 2)(5 \cdot 2) = 3^2 \cdot 5^2 \cdot 2^2$. That way, $\sqrt{15mn} = \sqrt{3^2 \cdot 5^2 \cdot 2^2} = 30$ is still an integer, making the least possible value of mn equal to $6 \cdot 10 = 60$.

8. B (Estimated Difficulty Level: 4)

A good opportunity to plug in real numbers! For example, suppose set M consists of the integers: 2, 4, 6, 8, 10, and 12. The sum of the least three is 12 and the sum of the greatest three is 30, so answer B is correct.

You say you want an algebraic solution? Suppose that n is the first even integer. The remaining integers are then $n+2$, $n+4$, $n+6$, $n+8$, and $n+10$. The sum of the least three of these integers is $x = n+(n+2)+(n+4) = 3n+6$, and the sum of the greatest three of these integers is $y = (n+6)+(n+8)+(n+10) = 3n+24$. So, $y-x = 18$, or $y = x + 18$.

9. 792 (Estimated Difficulty Level: 5)

To get the greatest difference, we want to subtract a small number from a large one, so we will need the digit 9 and the digit 1, in order to make a number in the 100's and a number in the 900's. The large number will look like $9N1$ and the small number will look like $1N9$, where N is a digit from 2 to 8. You will find that, no matter what you make N, the difference is 792.

10. E (Estimated Difficulty Level: 5)

Suppose that the integer is n. The result of subtracting n from its square is $n^2 - n = n(n-1)$, which is the product of two consecutive integers, so answer E is correct.

Notice that if you multiply any two consecutive integers, the result is always even, since it is the product of an even integer and an odd integer. To win an Erik The Red Viking Hat, see if you can determine why the result is never a negative integer.