1) How many unique 3 digit codes can be created from the 5 digits {1, 2, 3, 4, 5} if repeats are possible?

2) In how many ways can I create 2 letter words from the letters in the word DAVE? Any 2 letters form a 2 letter word.

3) Malcolm has to choose 5 numbers for his password (from digits 0 to 9) and repeat digits are not possible. How many different arrangements of numbers could Malcolm choose?

4) 10 students have appeared for a test in which the top three will get a prize. How many possible ways are there to get the prize winners?

5) How many unique passwords can be made from 4 letters if repeat letters are possible?

6) How many permutations of 3 vowels can be made from the vowels (a,e,i,o,u) without any repeat letters?

7) How many unique passwords can be made from 2 letters and 3 digits (0 to 9) if repeats are possible?

8) How many 4-digit even numbers can be created from the digits {0, 1, 2, 3}?

9) There are 6 people who will sit in a row but out of them, Ronnie, will always be left of Annie and Rachel will always be right of Annie. How many row arrangements are possible?

10) In how many ways can the name Saige be re-arranged such that the letter "i" comes to the right of the letter "g"?

1) How many unique 3 digit codes can be created from the 5 digits {1, 2, 3, 4, 5} if repeats are possible?

5 choices × 5 choices × 5 choices = 5^3 possible 3 digit codes including repeats

2) In how many ways can I create 2 letter words from the letters in the word DAVE? Any 2 letters form a 2 letter word.

Choose two letters from four = P(4,2) = 4!/2! = 4 choices × 3 choices = 12. Order matters i.e. DA is a different word than AD.

3) Malcolm has to choose 5 numbers for his password (from digits 0 to 9) and repeat digits are not possible. How many different arrangements of numbers could Malcolm choose?

P(10,5) = 10!/5! = 10 × 9 × 8 × 7 × 6

4) 10 students have appeared for a test in which the top three will get unique prizes. How many possible ways are there to get the prize winners?

P(10,3) = 10!/7! = 10 × 9 × 8

5) How many unique 3 letter passwords can be made from 4 letters if repeat letters are possible?

4 × 4 × 4 = 4^3

6) How many permutations of 3 vowels can be made from the vowels (a,e,i,o,u) without any repeat letters?

P(5,3) = 5!/2! = 5 × 4 × 3

7) How many unique passwords can be made from 2 letters and 3 digits (0 to 9) if repeats are possible?

Event (Choose 2 letters)   Event (Choose 3 digits)
26 × 26 = 26^2 arrangements of letters   10 × 10 × 10 = 10^3 arrangements of digits

Answers

Arrangements (Both) = # Ways Choose 2 letters × # Ways choose 3 digits
$= 26 \times 26 \times 10 \times 10 \times 10 = 26^2 \times 10^3$

8) How many 4-digit even numbers can be created from the digits {0, 1, 2, 3}?

Dependencies make some arrangements not possible. First the digit 0 cannot be used in the 1st position else the result is a 3-digit not 4-digit number. Also, the last digit must a 0 or 2 in order to be an even number.

Group the options for the 1st digit and then count the arrangements of the other 3 digits by working from the last digit backward.

1: 2 ways to choose last digit(0 or 2) × 2 ways to choose remaining × 1 way for last digit
  = **4** ways to start with a 1.

2: 2 ways to choose last digit(0 or 2) × 2 ways to choose remaining × 1 way for last digit
  = **4** ways to start with a 2.

3: 2 ways to choose last digit(0 or 2) × 2 ways to choose remaining × 1 way for last digit
  = **4** ways to start with a 3.

In total, that is 4 + 4 + 4 = 12 ways to create 4-digit even numbers.

9) There are 6 people who will sit in a row, but out of them Ronnie will always be left of Annie and Sanjay will always be right of Annie. How many row arrangements are possible?

If 3 positions are determined to be together then 3 names are reduced to essentially 1 group which makes a total of 1 group + 3 other names = 4 objects to choose for ordering. 4 unlike objects can be arranged in 4! ways.

10) In how many ways can the name Saige be re-arranged such that the letter "i" comes to the right of the letter "g"?

There are 5! ways to arrange 5 unlike letters. Among that number of arrangements exactly half will have i to the left of g and the other half will have i to the right of g. As a result, 5!/2 arrangements of the letters SAIGE exist with "i" to the right of "g".

1) In how many ways can Max select 3 toppings from 10 available toppings for his pizza?

2) In how many ways can coach Johnson select 5 young women for the basketball team if 15 players try out?

3) How many different committees of 5 people can be chosen from 10 people?

4) Jones is the Chairman of a committee. In how many ways can a committee of 5 be chosen from 10 people given that Jones must be one of them?

5) You just bought 3 tickets to the Yankees and you have 5 friends you could invite. How many different combinations of friends could you choose to join you at the game?

6) Any two lines in a plane form at most one intersection. If there are 10 distinct lines, then what is the maximum number of intersections possible?

7) From previous lessons we learned a line can be formed from any two points. If there are 5 points in a plane then what is the maximum number of unique lines that can be drawn between the points?

8) Six friends want to play enough games of chess to be sure every one plays everyone else. How many games will they have to play?

9) 16 teams enter a soccer competition. They are divided up into four Pools (A, B, C and D) of four teams each.

Every team plays one match against the other teams in its Pool.

After the Pool matches are completed:
• the winner of Pool A plays the second placed team of Pool B
• the winner of Pool B plays the second placed team of Pool A
• the winner of Pool C plays the second placed team of Pool D
• the winner of Pool D plays the second placed team of Pool C

The winners of these four matches then play semi-finals, and the winners of the semi-finals play in the final.

How many matches are played altogether?

10) A lottery ticket requires players to select 5 distinct numbers from 00 to 99 to appear on the ticket. How many unique tickets are possible?

1) In how many ways can Max select 3 toppings from 10 available toppings for his pizza? The order of choosing toppings does not matter so it is a combination not permutation solution.

C(10,3) = 10!/7!3! = (10 × 9 × 8)/(3 × 2 × 1) = 5 × 3 × 8

2) In how many ways can coach Johnson select 5 young women for the basketball team if 15 players try out? Order does not matter among selected team members.

C(15,5) = 15!/10!5! = (15 × 14 × 13 × 12 × 11)/(5 × 4 × 3 × 2 × 1) = 7 × 13 × 3 × 11

3) How many different committees of 5 people can be chosen from 10 people?
Order does not matter among selected committee members.
C(10,5) = 10!/5!5! = (10 × 9 × 8 × 7 × 6)/(5 × 4 × 3 × 2 × 1) = 3 × 2 × 7 × 6

4) Jones is the Chairman of a committee. In how many ways can a committee of 5 be chosen from 10 people given that Jones must be one of them?
If Jones is selected with certainty there is only 1 way to select Jones. Multiply 1 by the rest which becomes choose 4 out of 9 remaining people where order does not matter.

C(9,4) = 9!/5!4! = (9 × 8 × 7 × 6)/(4 × 3 × 2 × 1) = 3 × 7 × 6

5) You just bought 3 tickets to the Yankees and you have 5 friends you could invite. How many different combinations of friends could you choose to join you at the game? This is similar to the problem above. Do not forget you are certain to use 1 ticket so the question is really asking how many combinations of 2 tickets can be given to 5 different friends.

C(5,2) = 5!/3!2! = (5 × 4)/(2 × 1) = 5 × 2

6) Any two lines in a plane form at most one intersection. If there are 10 distinct lines, then what is the maximum number of intersections possible? The question is essentially asking how many distinct pair of lines can be created from 10 lines?

C(10,2) = 10!/8!2! = (10 × 9)/(2 × 1) = 5 × 9

7) From previous lessons we learned a line can be formed from any two points. If there are 5 points in a plane then what is the maximum number of unique lines that can be drawn between the points? This is similar to #6 above, but count pairs of points.

$C(5,2) = 5!/3!2! = (5 \times 4)/(2 \times 1) = 5 \times 2$

8) Six friends want to play enough games of chess to be sure every one plays everyone else. How many games will they have to play?

Again similar to above, but count pairs of players.

$C(6,2) = 6!/4!2! = (6 \times 5)/(2 \times 1) = 5 \times 3$

9) 16 teams enter a soccer competition. They are divided up into four Pools (A, B, C and D) of four teams each.

Every team plays one match against the other teams in its Pool.

After the Pool matches are completed:
• the winner of Pool A plays the second placed team of Pool B
• the winner of Pool B plays the second placed team of Pool A
• the winner of Pool C plays the second placed team of Pool D
• the winner of Pool D plays the second placed team of Pool C

The winners of these four matches then play semi-finals, and the winners of the semifinals play in the final.

How many matches are played altogether?

There are dependencies here that can be grouped and counted. First, each game is a pair of teams from 4 teams in a pool.

$C(4,2) = 4!/2!2! = (4 \times 3)/(2 \times 1) = 6$ games for every pool $\times$ 4 pools = **24** total pool games.

24 pool games + 4 quarter final games + 2 semifinal games + 1 final = 31 games total

10) A lottery ticket requires players to select 5 distinct numbers from 00 to 99 to appear on the ticket. How many unique tickets are possible? Order doesn't matter. The 5 winning numbers can be on the same ticket in any order.

$C(100,5) = 100!/95!5!$

1. Permutations

Permutation is a fancy word for ordered. In the previous lesson students learned n unlike objects can be ordered in n! ways. e.g.

Ways to line up 5 students (out of 5 students) = $5 \times 4 \times 3 \times 2 \times 1 = 5!$

There are 5 students available for the 1st choice, 4 left for the 2nd choice, 3 left for the 3rd choice etc. until all students are chosen. In general, when discussing permutations (and combinations), students should consider the cases where less than all students are chosen, but instead some subset r is chosen. e.g.

Ways to select or line up 2 students out of 5 students = $5 \times 4 = 5!/3!$

There are only 2 choices not 5 choices to make in this case so only two events are multiplied, but the approach is the same. In general, the formula for selecting or lining up r objects <u>where order of the selected items matter</u> from n available choices is...

$P(n,r) = n!/(n - r)!$

In the case above, the number of ways to choose 2 students out of 5 is $5!/(5 - 2)! = 5!/3! = 5 \times 4$. The results are the same. If all n students are chosen (r = n) this formula also reduces to $n!/0! = n!$ the same result from the previous lesson.

Students should gain comfort with the logical approach as well as the formula itself. The logical approach of identifying how many ways the 1st choice can be made then multiplying by the number of options for the 2nd choice and so on can be more flexible in application when certain features of the question are varied.

2. Combinations

Combinations are similar to permutations except <u>order of the selected items does not matter</u>.

\# Ways to select 2 students out of 5 students = $(5 \times 4)/2! = 5!/3!2!$

Again, there are only 2 choices not 5 choices to make so only two events are multiplied when choosing $r = 2$ objects out of $n = 5$ objects. The key difference is that the order of the chosen objects does not matter. Instead of 2! ways to arrange AB and BA assuming the choices were A or B, there is only one way to arrange the <u>selected items because AB is the same as BA for the purposes of the problem</u>. As a result, to avoid overcounting r! arrangements when only 1 exists, the student must divide by r! in combinations. That is the only difference between the permutation and combination formula. In general, the formula for selecting r objects where <u>order of the selected items does not matter</u> from n available choices is...

$$C(n,r) = n!/(n - r)!r!$$

In the case above, the number of ways to choose 2 students out of 5 is $5!/(5 - 2)!2! = 5!/3!2! = (5 \times 4)/2 = 10$. Note if all n students are chosen ($r = n$) this formula reduces to $n!/0!n! = 1$ as expected. There is only one way to arrange objects that are all the same.

How do I know if order matters or not?

The key question every student should ask themselves in each permutation/combination problem is whether order of the selected items matters? Unfortunately, this requires some reasoning to determine based on the conditions of the problem, but experience will help substantially and a few examples are shown below to highlight the differences.

Examples of permutation where order of selected items matters:
1. Choosing digits for a unique ID or number. 325 is a different number or ID than 523 even though the digits chosen are the same.
2. Lining objects up or awarding medals for the first few positions. 1st position is different than 2nd position and so on.

Examples of combination where order of selected items does not matter:
1. Choosing players for the team. It doesn't matter if Henry is chosen 1st and Malcolm 2nd. For the purposes of the problem they are both on the team and therefore alike.
2. Ordering toppings on a pizza. It does not matter if pepperoni is chosen before mushrooms or not. If they are both ordered then the pizza will be the same regardless of what order the toppings were selected.